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H-CLOSURE FOR BITOPOLOGICAL SPACES

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1. Introduction

The purpose of this paper is to describe a type of H-closure for bitopological spaces which parallels the pairwise compactness of Fletcher, Hoyle and Patty [2]. Connections will be established between pairwise (p.w.) H-closure and topological θ-convergence and their relationship to the semiregularization of a topological space. Brief descriptions will be given of p.w. light compactness and a method of constructing a p.w. H-closure of a bitopological space.

The definitions used herein are standard and listed for convenience. A family of subsets $\{U_i\}$ of a bitopological space (X, P, Q) is *p.w. open* if $\{U_i\}$ contains at least one non-dense member from each of P and Q and *p.w. closed (regular closed)* if it contains at least one closed (regular closed) subset from each topology. A filter F on X is *p.w. open* if F contains a p.w. open subcollection and is *completely p.w. open* if it contains a p.w. open subcollection which generates the filter. Each p.w. open (completely p.w. open) filter is contained in a filter which is maximal among the family of p.w. open (completely p.w. open) filters. An *infinite sequence on X is p.w. open* if there exists a positive integer N and a non-dense P-open set U and non-dense Q-open set V with $a_n \in U \cap V$ for all $n > N$. The space (X, P, Q) is *p.w. completely Hausdorff* if for any distinct points x and y of X, there exists a P-open neighborhood U of x and a Q-open neighborhood V of y with $\mathrm{cl}_P(U) \cap \mathrm{cl}_Q(V) = \phi$.

In a topological space, a filter F *θ-converges to x* (*θ-accumulates at x*) if $\mathrm{cl}(V) \in F$ ($\mathrm{cl}(V) \cap A \neq \phi$ for all $A \in F$) for any open neighborhood V of x.

All terms not previously described which are prefaced by pairwise (p.w.) are understood to mean that the stated property holds with respect to each topology. The term space signifies a nondegenerate bitopological space.

2. Pairwise H-closed spaces

DEFINITION 2.1. A bitopological space (X, P, Q) is *pairwise H-closed (p.w.H.)* if for every pairwise open cover of X there exists a finite subcollection whose closures (in their respective topologies) cover X.

The proofs of the first three propositions are direct and will be omitted.

PROPOSITION 2.2. *If (X, P, Q) is p.w.H. and P and Q are regular topologies then (X, P, Q) is pairwise compact.*

PROPOSITION 2.3. *The pairwise continuous image of a p.w.H. space is p.w.H.*

PROPOSITION 2.4. *Let (X, P, Q) be p.w.H. If A is a P-regular closed subset of X, then A is Q-H-closed.*

EXAMPLE 2.5. A simple example of a p.w. Hausdorff, p.w.H., non p.w. compact space can be constructed by: on the nonzero reals, let P be the topology generated by $\{\{x\} | x>0\}$ together with all intervals of the form (x, ∞) for $x<0$ while Q is generated by $\{\{x\}, x<0\}$ and $(-\infty, x)$ for $x>0$.

The semiregularization θP of a topological space (X, P) is that topology on X having as a basis all P-regular open sets (see [1]). By λP we will denote the finest topology on X such that if F is an ultrafilter which P-θ-converges to x then F λP-converges to x. λP will be shown to be related to products of p.w.H. spaces (Proposition 2.14).

LEMMA 2.6. *For any topology P on X, λP is coarser than θP.*

PROOF. Suppose the ultrafilter F θP-converges to x. If V is a P-open neighborhood of x then $\text{int}_p(\text{cl}_p V)$ is a regular open neighborhood of x so $\text{int}_p(\text{cl}_p V) \in F$ and $\text{cl}_p V \in F$ so F θ-converges to x and λP is coarser than θP.

PROPOSITION 2.7. *For any topology P on X, $\lambda P = \theta P$ if and only if θP is regular.*

PROOF. From Lemma 2.6 it suffices to show regularity equivalent to the condition θP coarser than λP so assume F is an ultrafilter which P-θ-converges to x and that θP is regular. If V is a P-regular open neighborhood of x there exists a P-regular open neighborhood U of x with $U \subset \text{cl}(U) \subset V$ and $\text{cl}(V) \in F$ which implies F θP-converges to x.

Conversely, if θP is coarser than λP, let $x \in X$ and F_x be the filter generated by $\{\text{cl}_P(V) | x \in V, V\ P\text{-open}\}$. Since F_x P-θ-converges, F_x also θP-converges to x so if V is a P regular open neighborhood of x there exists a P-open neighborhood of x with $x \in U \subset \text{cl}_P(U) \subset V$. But $W = \text{int}_P(\text{cl}_P U)$ is P-regular open and $x \in W \subset \text{cl}_{\theta P}(W) = \text{cl}_P(W) \subset V$ so $(X, \theta P)$ is regular.

An example of a space whose semiregularization is not regular is the Arens square [7].

REMARK. Before demonstrating the existence of a filter characterization for p. w. H. spaces some observations on maximal p. w. open filters will be essential.

1. A filter is p. w. open if and only if it contains a non-dense regular open set from each topology.

2. If an ultrafilter F is not p. w. open, say F contains no non-dense P-regular open set then F must contain all P-regular closed sets.

3. An ultrafilter F does not contain a non-dense P-regular open set if and only if F P-θ-converges to every point.

4. If P and Q are completely Hausdorff topologies on X then each ultrafilter is a maximal p. w. open ultrafilter.

Similar to the situation for H-closed topological spaces [3], filter characterizations exist for p. w. H. spaces.

PROPOSITION 2.8. *The following statements about a bitopological space* (X, P, Q) *are equivalent*:

i) (X, P, Q) *is p. w. H.*

ii) *Any p. w. family of regular closed sets whose interiors have the finite intersection property (f. i. p.) has a non-empty intersection.*

iii) *Every maximal p. w. open filter p. w. θ-converges.*

iv) *Every maximal completely p. w. open filter p. w. converges.*

PROOF. (i)→(ii). Assume $\{F_i\}$ is a p. w. family of regular closed sets whose interiors have the f. i. p. and $\bigcap F_i=\phi$. Then $X\subset\bigcup(X-F_i)$ but $X\not\subset\bigcup \mathrm{cl}(X-F_j)$ $=\bigcup X-\mathrm{int}F_j$ for any finite subset so (X, P, Q) is not p. w. H.

(ii)→(iii). If F is a maximal p. w. open filter which does not p. w. θ-converge then for each point x there exists either a P-open or a Q-open set V_x with $x\in V_x\subset \mathrm{cl}(V_x)\notin F$ (closure in the respective topology). Then $X-\mathrm{cl}(V_x)\in F$ is regular open so $\{\mathrm{cl}(X-\mathrm{cl}V_x)\}$ is a regular closed family whose interiors have the f. i. p. but which has empty intersection. By Remark 3, the family $\{\mathrm{cl}(X-\mathrm{cl}V_x)\}$ can be chosen as a p. w. regular closed family so the conclusion follows.

iii)→i) Suppose (X, P, Q) is not p. w. H. and $\{V_i\}$ is a p. w. open cover with no finite subfamily whose closures cover X. Then $\{X-\mathrm{cl}(V_i)\}$ is a p. w. open collection with f. i. p. but any maximal p. w. open filter containing this collection cannot p. w. θ-converge to any point. The equivalence of (i) and (iv) is proved similarly.

By Remark 3 and Proposition 2.8, one can note that a bitopological space is p.w.H. if and only if every ultrafilter p.w. θ-converges or θ-converges to no point with respect to one topology and to every point with respect to the other topology.

COROLLARY 2.9. *If* (X, P, Q) *is p.w.H. and* (X, P) *is Hausdorff but not H-closed then Q cannot be completely Hausdorff.*

PROOF. Since (X, P) is not H-closed, some ultrafilter F must fail to P-θ-converge (see [3]). From Remark 4, F must Q-θ-converge to each point and hence F contains the closures of all Q-open sets so Q cannot be completely Hausdorff.

If a topological space is *extremely connected* whenever no two open sets have disjoint closures, then:

COROLLARY 2.10. *Under the hypothesis of Corollary 2.9, Q is extremely connected.*

COROLLARY 2.11. *If* (X, P) *is completely Hausdorff and H-closed and Q is any topology on X for which* (X, P, Q) *is p.w. completely Hausdorff and p.w. H-closed then* $\lambda P = \lambda Q$.

PROOF. If F is an ultrafilter which P-θ-converges to x then (X, P) completely Hausdorff and p.w. Hausdorff implies F Q-θ-converges to x.

Conversely, if F Q-θ-converges to x, P H-closed implies F P-θ-converges and (X, P, Q) p.w. completely Hausdorff implies F P-θ-converges to x.

The next corollary is analogous to a result for p.w. compact spaces [2].

COROLLARY 2.12. *If* (X, P, Q) *is p.w. completely Hausdorff and* (X, P) *and* (X, Q) *are H-closed then* $\lambda P = \lambda Q$.

PROOF. Let F be an ultrafilter which θ-converges to x with respect to P. Then Q is H-closed so F must Q-θ-converge and the space being completely Hausdorff implies F Q-θ-converges to x.

In Hausdorff topological spaces it is well known that H-closure is equivalent to every continuous image of the space in any Hausdorff space is closed. A partial result of similar type exists for p.w.H. spaces. The direct proof will be omitted.

PROPOSITION 2.13. *If (X, P, Q) is p.w.H., (Y, R, S) is p.w. Hausdorff and $f: (X, P, Q) \to (Y, R, S)$ is p.w. continuous, then the image of f is closed in the coarsest topology on Y generated by $R \cup S$.*

Products of p.w. compact spaces were classified in [6]. A corresponding result holds for p.w. H-spaces using the filter characterizations of H-closure of Long and Herrington [3]. P and Q represent the appropriate product topologies.

PROPOSITION 2.14. *If $\{(X_i, P_i, Q_i) | i \in I\}$ is a family of bitopological spaces with $|I| > 1$, $(\Pi X_i, P, Q)$ is p.w.H. if and only if each (X_i, P_i, Q_i) is p.w.H. and if some (X_j, P_j) $((X_j, Q_j))$ is not H-closed then $(X, \lambda P_i)$ $((X, \lambda Q_i))$ is discrete for $i \neq j$.*

PROOF. The product space being p.w.H. implies, by Proposition 2.3, that each factor space is p.w.H.

Suppose some (X_j, P_j) is not H-closed. Then some ultrafilter F_j on X_j does not θ-converge [3]. If, for each $k \neq j$, F_k is an ultrafilter on X_k and G is an ultrafilter on ΠX_i whose projections $p_i(G)$ are the F_i then G does not θ-converge with respect to P [3] so must θ-converge to every point of ΠX_i with respect to Q and each λQ_i is indiscrete.

Conversely, if F is an ultrafilter on ΠX_i with no P-θ-limit points then some $p_i(F)$ does not P_i-θ-converge and (X, P_i) is not H-closed. But (X_i, P_i, Q_i) p.w.H. and $(X_j, \lambda Q_j)$ indiscrete for $i \neq j$ implies F must Q-θ-converge to every point. If F both P and Q-θ-converges, since each (X_i, P_i, Q_i) is p.w.H., each projection $p_i(F)$ has a p.w. θ-limit in (X_i, P_i, Q_i) so F must p.w. θ-converge.

COROLLARY 2.15. *In the category of p.w. completely Hausdorff spaces, if $|I| > 1$, $(\Pi X_i, P_i, Q_i)$ is p.w. H-closed if and only if each $\lambda P_i = \lambda Q_i$ and each P_i and Q_i are H-closed.*

PROOF. If the product space is p.w.H and some $P_i(Q_i)$ is not H-closed then each $\lambda Q_j(\lambda P_j)$ is indiscrete for $j \neq i$ and (X_j, P_j, Q_j) is not p.w. completely Hausdorff. Since each (X_i, P_i) and (X_j, Q_j) are H-closed, Corollary 2.12 implies $\lambda P = \lambda Q$.

3. Pairwise light compactness

Similar to the relationship between H-closure and light compactness for topological spaces exists a corresponding relationship for bitopological spaces.

DEFINITION 3.1. A space (X, P, Q) is *p.w. lightly compact* if for every countable p.w. open cover of X there exists a finite subcollection whose closures cover X.

The subsequent two propositions are similar to Propositions 2.3. and 2.8.

PROPOSITION 3.2. *The continuous image of a lightly compact space is lightly compact.*

PROPOSITION 3.3. *The following are equivalent:*

i) (X, P, Q) *is p.w. lightly compact.*

ii) *Every countable p.w. family of regular closed sets whose interiors having the f.i.p. has a non-empty intersection.*

iii) *Every p.w. open filter with countable filterbase has a p.w. θ-accumulation point.*

iv) *Every p.w. open infinite sequence has a p.w. θ-accumulation point.*

REMARK. If (X, P, Q) is p.w. lightly compact and F has a countable filterbase and no P-θ-accumulation point, then F Q-θ-accumulates at every point.

The proof of the next proposition is similar to Proposition 2.14.

PROPOSITION 3.4. *If $\{(X_i, P_i, Q_i) | i \in I\}$ is a countable family of bitopological spaces with $|I| > 1$ whose product space is p.w. first countable (ie, both topologies first countable) then $(\Pi X_i, P, Q)$ is p.w. lightly compact if and only if each factor space is p.w. lightly compact and if some (X_j, P_j) $[(X_j, Q_j)]$ is not lightly compact then λX_i is indiscrete for $i \neq j$.*

4. Pairwise H-closure of a bitopological space

Liu [5] constructed an absolute closure of a Hausdorff topological space based on the absolute closure of Katetov [4]. For bitopological spaces an H-closure can be constructed which is nearly a facsimile of Liu's construction.

DEFINITION 4.1. A p.w. H-closed space $(\hat{X}, \hat{P}, \hat{Q})$ is the *p.w. H-closure* of a space (X, P, Q) if there exists a p.w. embedding $i: X \to \hat{X}$ whose image is p.w. dense in $\hat{X}$.

The proofs of the following lemma and the proposition will not be included as they closely resemble the corresponding proofs for the topological case.

LEMMA 4.2. i) *If X is a p.w. open, p.w. dense subset of a space Y and F is a maximal completely p.w. open filter on Y, then $F'=F\cap X=\{A\cap X \mid A\in F\}$ is a maximal completely p.w. open filter on the subspace X.*

ii) *F' on X p.w. converges to x iff F on Y p.w. converges to x.*

PROPOSITION 4.3. *For any space (X, P, Q) there exists a space $(\hat{X}, \hat{P}, \hat{Q})$ which is a p.w. H-closure of X such that the topology on $\hat{X}$ generated by $\hat{P}\cup\hat{Q}$ is Hausdorff if X is p.w. Hausdorff.*

PROOF. Let $\tilde{X}$ be the set of all non-p.w. convergent maximal completely open filters on X and $\hat{X}=X\cup\tilde{X}$. Describe two topologies $\hat{P}$ and $\hat{Q}$ on $\hat{X}$ as follows:

A basis for $\hat{P}(\hat{Q})$ consists of all $B=U\cup\{F\}$ where $U\in P(U\in Q)$, $U\in F$ $F\in\tilde{X}$. This construction and the subsequent proof using Lemma 4.2 parallel Liu's construction of an absolute closure for Hausdorff topological spaces.

The question remains as to suitable conditions under which the p.w. H-closure of Proposition 4.3 is p.w. Hausdorff and whether a suitable description can be given of those spaces for which the given construction is the largest p.w. H-closure.

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