Kyungpook Math. J.
Volume 18, Number 2
December, 1978

ON THE MANN ITERATION PROCESS IN A UNIFORMLY CONVEX BANACH SPACE

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Recently many papers concern about the existence of a fixed point for a self mapping. For example, F.E.Browder [1], W.A.Kirk [2] and D.Ghöde [3] have proved that every nonexpansive mapping T of a closed bounded convex subset K of a uniformly convex Banach space into K has a fixed point in K. For a self mapping T of a compact interval of the real line having a unique fixed point, Mann [4] proved that the iteration process $v_{n+1}=(1-d_n)v_n+d_nTv_n$ converges to the fixed point, for some sequence $\{d_n\}$ in $(0,1)$. Groetsch [5] generalized the procedure for nonexpansive mappings on uniformly convex Banach spaces. In this paper we give some sufficient conditions to ensure that every iteration process, starting from any point, converges to some fixed point for a quasi-non-expansive self mapping T.

Let X be a uniformly convex Banach space and let δ be the modulus of convexity of X, that is, δ is a function mapping $[0,2]$ into $[0,1]$ which is defined as follows:

$$\delta(\varepsilon)=\inf\left\{1-\left\|\frac{x+y}{2}\right\| : x,y\in X,\ \ \|x\|\leqslant 1,\ \ \|y\|\leqslant 1,\ \ \|x-y\|\geqslant\varepsilon\right\}$$

It is easy to see that δ is strictly increasing and continuous such that $\delta(0)=0$ and $\delta(2)=1$. Moreover the inequalities

$$\|x\|\leqslant d,\ \|y\|\leqslant d \text{ and } \|x-y\|\geqslant\varepsilon \text{ imply } \left\|\frac{x+y}{2}\right\|\leqslant\left[1-\delta\left(\frac{\varepsilon}{d}\right)\right]d.$$

THEOREM 1. *Suppose that C is a convex subset of a uniformly convex Banach space X and $T: C\longrightarrow C$ is a mapping with $F(T)\neq\phi$ such that $\|Tx-p\|\leqslant\|x-p\|$ for all $x\in C$ and $p\in F(T)$, where $F(T)$ is the set of all fixed points of T. Let $\{d_n\}$ be a sequence from $(0,1)$ such that there exists a subsequence $\{d_{n_k}\}$ which converges to $d\in(0,1)$. Then for each $v\in C$, the iterated sequence $\{v_n\}$ which is defined by $v_1=v$ and $v_{n+1}=(1-d_n)v_n+d_nTv_n$ satisfies that lim inf $\|Tv_n-v_n\|=0$.*

PROOF. By the convexity of C, we see that $v_n\in C$ for all $n=1,2,\cdots$ and then

$\|Tv_n-p\| \leqslant \|v_n-p\|$. It is due to the definition of $\{v_n\}$, we have

$\|v_{n+1}-p\|=\|(1-d_n)v_n+d_nTv_n-p\|=\|(1-d_n)(v_n-p)+d_n(Tv_n-p)\| \leqslant (1-d_n)\|v_n-p\| +d_n\|v_n-p\|=\|v_n-p\|$.

Thus $\{\|v_n-p\| : n=1,2,\cdots\}$ is a decreasing sequence, and so

$$\lim \|v_n-p\|=\inf \|v_n-p\|=M.$$

We prove our result by the following two cases:

Case 1. If $\lim \|v_n-p\|=M=0$, then by the inequalities

$$\|Tv_n-v_n\| \leqslant \|Tv_n-p\|+\|p-v_n\| \leqslant 2\|v_n-p\|,$$

we obtain that $\lim\inf \|Tv_n-v_n\|=\lim \|Tv_n-v_n\|=0$.

Case 2. If $\lim \|v_n-p\|=\inf \|v_n-p\|=M \neq 0$, we also claim that $\lim\inf \|Tv_n-v_n\|=0$. Suppose not, there is an $\varepsilon>0$ and a positive integer N_1 such that $\|Tv_n-v_n\| \geqslant \varepsilon$ for all $n \geqslant N_1$. For such an n we have $\varepsilon \leqslant \|Tv_n-v_n\| \leqslant 2\|v_n-p\|$. Hence $\varepsilon \leqslant 2M$. Now from the hypothesis that $0<d<1$, there is an $\eta>0$ such that $\left|d-\frac{1}{2}\right|<\eta<\frac{1}{2}$. Then we have that $0<\frac{\varepsilon(1-2\eta)}{M}<2$, this implies that $0<1-\delta(\varepsilon(1-2\eta)/M)<1$. By the fact that the function $t(1-\delta(\varepsilon(1-2\eta)/Mt))$ is coutinuous and strictly increasing in $t\in[1,\infty)$ and $0<1-\delta(\varepsilon(1-2\eta)/M)<1$, there is a $c>0$ such that $(1+c)(1-\delta(\varepsilon(1-2\eta)/M(1+c)))<1$. Since $\lim \|v_n-p\|=\inf \|v_n-p\|=M$ and $d_{n_k} \longrightarrow d$, there is a positive integer $N_2 \geqslant N_1$ such that for all $n \geqslant N_2$, $n_k \geqslant N_2$ we have

$$M \leqslant \|v_n-p\|<M(1+c) \text{ and } \left|d_{n_k}-\frac{1}{2}\right|<\eta.$$

It follows from the definition of $\{v_n\}$ that

$$\|v_{n_k+1}-p\|=\|(1-d_{n_k})(v_{n_k}-p)+d_{n_k}(Tv_{n_k}-p)\|$$

and we may represent $\|(1-d_{n_k})(v_{n_k}-p)+d_{n_k}(Tv_{n_k}-p)\|$ into the following two different forms which depend on $d_{n_k} \geqslant \frac{1}{2}$ or $d_{n_k}<\frac{1}{2}$.

i) $d_{n_k} \geqslant \frac{1}{2}$ and $n_k \geqslant N_2$, we have

$\|(1-d_{n_k})(v_{n_k}-p)+d_{n_k}(Tv_{n_k}-p)\|=\|2(1-d_{n_k})(v_{n_k}-p)+(2d_{n_k}-1)(Tv_{n_k}-p)+Tv_{n_k}-p\|/2$

Set $x=2(1-d_{n_k})(v_{n_k}-p)+(2d_{n_k}-1)(Tv_{n_k}-p)$ and $y=Tv_{n_k}-p$,

then $\|x\| \leqslant 2(1-d_{n_k})\|v_{n_k}-p\|+(2d_{n_k}-1)\|Tv_{n_k}-p\| \leqslant 2(1-d_{n_k})\|v_{n_k}-p\|+(2d_{n_k}-1)\|v_{n_k}-p\| =\|v_{n_k}-p\|<M(1+c)$,

$\|y\|=\|Tv_{n_k}-p\|\leqslant\|v_{n_k}-p\|<M(1+c)$ and $\|x-y\|=\|2(1-d_{n_k})(v_{n_k}-p)+(2d_{n_k}-2)(Tv_{n_k}-p)\|=\|2(1-d_{n_k})(v_{n_k}-Tv_{n_k})\|=2(1-d_{n_k})\|v_{n_k}-Tv_{n_k}\|\geq(1-2\eta)\varepsilon$

ii) $d_{n_k}<\frac{1}{2}$ and $n_k\geqslant N_2$, we have $\|(1-d_{n_k})(v_{n_k}-p)+d_{n_k}(Tv_{n_k}-p)\|$

$$=\|(v_{n_k}-p)+(1-2d_{n_k})(v_{n_k}-p)+2d_{n_k}(Tv_{n_k}-p)\|/2.$$

Set $\quad x_1=v_{n_k}-p$ and $y_1=(1-2d_{n_k})(v_{n_k}-p)+2d_{n_k}(Tv_{n_k}-p)$,

then we have

$$\|x_1\|=\|v_{n_k}-p\|<M(1+c),$$

$$\|y_1\|\leqslant(1-2d_{n_k})\|v_{n_k}-p\|+2d_{n_k}\|Tv_{n_k}-p\|\leqslant(1-2d_{n_k})\|v_{n_k}-p\|+2d_{n_k}\|v_{n_k}-p\|=\|v_{n_k}-p\|<M(1+c)$$

and $\|x_1-y_1\|=\|2d_{n_k}(v_{n_k}-p)-2d_{n_k}(Tv_{n_k}-p)\|=\|2d_{n_k}(v_{n_k}-Tv_{n_k})\|\geqslant2d_{n_k}\varepsilon>(1-2\eta)\varepsilon.$
Hence for $n_k\geqslant N_2$, these imply that

$$\|v_{n_k+1}-p\|\leqslant(1-\delta((1-2\eta)\varepsilon/M(1+c)))(1+c)M<M.$$

It is a contradiction to the fact that inf $\|v_n-p\|\geqslant M$.

Hence lim inf $\|Tv_n-v_n\|=0$ in any case.

REMARK 1. In theorem 1, if $\{d_n\}$ converges to $d\in(0,1)$, then we can conclude that lim $\|Tv_n-v_n\|=0$. For if not, there are an $\varepsilon>0$ and a strictly increasing sequence $\{n_k\}$ of positive integers such that $\|Tv_{n_k}-v_{n_k}\|\geqslant\varepsilon$ for all $k=1,2,\cdots$. Then it follows from the same argument as we used in the proof of theorem 1, we have $\|v_{n_k+1}-p\|<M$ for sufficiently large k. This is a contradiction.

REMARK 2. Theorem 1 is still valid if we assume only that for some $p\in F(T)$ (instead of "for all $p\in F(T)$")

$$\|Tx-p\|\leqslant\|x-p\| \text{ for all } x\in C.$$

COROLLARY 1. *Suppose that X, C, $\{d_n\}$ and T satisfy the hypotheses of theorem* 1. *Suppose further that C is closed and if for any bounded set $K\subset C$ with $0\in\overline{(I-T)K}$ imply $0\in(I-T)\overline{K}$. Then for each $v_1\in C$, the iterated sequence $\{v_n\}$ which is defined as in theorem* 1 *converges to a fixed point of T.*

PROOF. By theorem 1, there is a subsequence $\{Tv_{n_k}-v_{n_k}\}$ which converges to 0, that is, $0\in\overline{(I-T)\{v_{n_k}\}}$. It follows from the hypothesis, we have $0\in(I-T)\overline{\{v_{n_k}\}}$. Hence there is a p in $\overline{\{v_{n_k}\}}$ such that $(I-T)p=0$, that is, $p\in F(T)$.

Since $p\in\overline{\{v_{n_k}\}}$, there is a subsequence $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$ such that $v_{n_{k_j}}\longrightarrow p$. We have known that $\{\|v_n-p\| : n=1, 2, \cdots\}$ is a decreasing sequence and therefore $\lim v_n=p$.

COROLLARY 2. *Suppose that C is closed convex subset of X. Suppose that T satisfies the hypothesis of theorem* 1 *and d_n converges to d, $0<d<1$. Suppose p is a cluster point of $\{v_n\}$ and T is continuous at p. Then v_n converges to p and $p\in F(T)$.*

PROOF. Since p is a cluster point of $\{v_n\}$, there is a subsequence $\{v_{n_j}\}$ which converges to $p\in C$. T is continuous at p, then we have $Tv_{n_j}\longrightarrow Tp$. It follows that $Tv_{n_j}-v_{n_j}\longrightarrow Tp-p$. By Remark 1, $Tv_n-v_n\longrightarrow 0$, hence $Tp-p=0$, that is, $p\in F(T)$. It follows from the fact that $\{\|v_n-p\| : n=1, 2, \cdots\}$ is decreasing and $v_{n_j}\longrightarrow p$, we have $\lim v_n=p$.

DEFINITION. (Kuratowski, [6]) Let X be a metric space. We define χ as a nonnegative valued function on the set of all bounded subsets of X such that

$\chi(D)=\inf \{r>0 : D$ is covered by finitely many sets with diameter $r\}$

It is well-known that $\chi(D)=0$ if and only if $\overline{D}$ is compact.

DEFINITION. Let X be a metric space, $C\subset X$. A continuous mapping T of C into X is said to be *a condensation mapping* if for each bounded subset D of C, TD is bounded and

$$\chi(TD)<\chi(D) \text{ for all } \chi(D)\neq 0$$

COROLLARY 3. *Suppose that C is a closed convex subset of a uniformly convex Banach space X. Suppose that T is a condensation mapping of C into C which satisfies the hypothesis of theorem* 1. *Then $\{v_n\}$ converges to a fixed point of T.*

PROOF. By theorem 1, we have $\liminf \|Tv_n-v_n\|=0$. Hence there is a subsequence $\{v_{n_k}\}$ such that $\|Tv_{n_k}-v_{n_k}\|$ converges to 0. We claim that $\overline{\{v_{n_k}\}}$ is compact. For if not, by the definition of condensation mapping, we have

$$\chi(\{Tv_{n_k} : k=1, 2, \cdots\})<\chi(\{v_{n_k} : k=1, 2, \cdots\}).$$

Hence there is some $\varepsilon>0$ such that

$$\chi(\{Tv_{n_k} : k=1, 2, \cdots\})+\varepsilon<\chi(\{v_{n_k} : k=1, 2, \cdots\})$$

Since $\lim \|Tv_{n_k}-v_{n_k}\|=0$, there is a positive integer N such that $\|Tv_{n_k}-v_{n_k}\|<\frac{\varepsilon}{3}$

for all $k \geqslant N$. Hence we have

$$\chi(\{Tv_{n_k} : k=1,2,\cdots\}) = \chi(\{Tv_{n_k} : k=N,N+1,\cdots\})$$

$$\geqslant \chi(\{v_{n_k} : k=N,N+1,\cdots\}) - \frac{2\varepsilon}{3} = \chi(\{v_{n_k} : k=1,2,\cdots\}) - \frac{2\varepsilon}{3}$$

$$> \chi(\{Tv_{n_k} : k=1,2,\cdots\}) + \frac{\varepsilon}{3}.$$

This is a contradiction. Thus there is a subsequence $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$ which converges to some point p. By the continuity of T, we have $Tv_{n_{k_j}} \longrightarrow Tp$. It follows that $Tv_{n_{k_j}} - v_{n_{k_j}} \longrightarrow Tp - p = 0$, that is, $p \in F(T)$. Since $\{\|v_n - p\| : n=1,2,\cdots\}$ is a decreasing sequence, hence $\lim v_n = p$.

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