Kyungpook Math. J.
Volume 18, Number 2
December, 1978

A NOTE ON PURE STATES AND STRICTLY PURE STATES OF BANACH *-ALGEBRAS

By Ilbong Jung

1. Introduction

Let A be a Banach *-algebra. In this paper, we consider the following questions which were raised by B. A. Barnes in [1].

Question 1. If α is a pure state of A, then is K_α a maximal left ideal of A ?

Question 2. If α and β are pure states of A and $K_\alpha = K_\beta$, does $\alpha=\beta$?

B. A. Barnes obtained in [1] that the strictly pure state have partial solutions of Question 1 and Question 2. We show in § 3 that the pure states have partial solutions of these questions. Also, we consider the similarity of π_α in § 4.

2. Preliminaries and Notations

Throughout this paper $(A, \|\cdot\|)$ denotes a Banach *-algebra. $\mathscr{H}$ is always a Hilbert space and $\mathscr{B}(\mathscr{H})$ is the algebra of all bounded operators on $\mathscr{H}$. Let α be a positive functional on A, and let

$$M(\alpha)=\sup\{|\alpha(a)|^2/\alpha(a^*a) : a\in A,\ \alpha(a^*a)\neq 0\}.$$

$\mathscr{P}$ is the set of all positive functionals α on A with the properties $\alpha(a^*)=\overline{\alpha(a)}$ for all $a\in A$ and $M(\alpha)<+\infty$. $\mathscr{P}_1$ is the set of all $\alpha\in\mathscr{P}$ with $M(\alpha)\leq 1$. For $\alpha\in\mathscr{P}$, K_α denotes the left kernel of α. The quotient space A/K_α is a pre-Hilbert space in the inner product $(a+K_\alpha, b+K_\alpha)=\alpha(b^*a)$. Let $\mathscr{H}_\alpha$ denotes the Hilbert space which is the completion of this pre-Hilbert space. A *-representation $a\longrightarrow \pi_\alpha(a)$ of A on $\mathscr{H}_\alpha$ is constructed by $\pi_\alpha(a)(b+K_\alpha)=ab+K_\alpha$, for all $a\in A$ and $b\in A$. Then $\pi_\alpha(a)$ is a bounded operator on A/K_α which extends uniquely to a bounded operator on $\mathscr{H}_\alpha$. $\alpha\in\mathscr{P}$ is a pure state of A if and only if $M(\alpha)=1$ and π_α is irreducible on $\mathscr{H}_\alpha$ (4, Theorem 21.34). $\alpha\in\mathscr{P}$ is a *strictly pure state* of A if α is a pure state of A and $a\longrightarrow\pi_\alpha(a)$ is strictly irreducible on $\mathscr{H}_\alpha$. α is a strictly pure state of A if and only if A/K_α is complete in the norm $|a+K_\alpha|_2 =\alpha(a^*a)^{\frac{1}{2}}$ if and only if $|\cdot|_2$ and $\|\cdot\|_q$ are equivalent, where $\|\cdot\|_q$ is the quotient norm on A/K_α (1, Theorem 2.1). The terminologies in this paper are due to [1] and [4].

3. Pure states and their left kernel

In [1], B. A. Barnes proved that if α is a strictly pure state of A, then K_α is a modular maximal left ideal of A. Also, he raised an open Question 1 in [1]. Proposition 3.1 is a generized partial solution of this problem.

PROPOSITION 3.1. *Let α be a pure state of A. Assume that there exists $K>0$ such that for all $a\in A$, $K\|a+K_\alpha\|_q\leq|a+K_\alpha|_2$. Then K_α is a modular maximal left ideal of A.*

PROOF. We define a functional $\tilde{\alpha}$ on A/K_α by $\tilde{\alpha}(a+K_\alpha)=\alpha(a)$. Since $M(\alpha)=1$, $\|\tilde{\alpha}\|=1$. By Hahn Banach Theorem, $\tilde{\alpha}$ can be extended to a bounded linear functional β on $\mathscr{H}_\alpha$ so that $\|\beta\|=\|\tilde{\alpha}\|$. Let z be an element of $\mathscr{H}_\alpha$ such that $\beta(x)=(x,z)$ for all $x\in\mathscr{H}_\alpha$, where $(\cdot,\cdot)$ is an inner product in the Hilbert space $\mathscr{H}_\alpha$. Then there exists an sequence $\{a_n\}$ in A such that $\{a_n+K_\alpha\}$ converges to z with respect to the Hilbert space norm. Since $\{a_n+K_\alpha\}$ is a $\|\cdot\|_q$-Cauchy sequence, there exists $v\in A$ such that $\|(a_n-v)+K_\alpha\|_q\longrightarrow 0$. Hence there exists a sequence $\{k_n\}$ in K_α such that $\|(a_n-v)+k_n\|\longrightarrow 0$. Since $\alpha(v^*(a_n-v))=\alpha(v^*(a_n-v)+v^*k_n)\longrightarrow 0$, $|\alpha(v^*v-v^*)|=|\lim(a_n+K_\alpha,\ v+K_\alpha)-(z,\ v+K_\alpha)|=0$. Hence $\alpha(v*v)=\alpha(v)$. i.e. $(v+K_\alpha,\ v+K_\alpha)=(v+K_\alpha,\ z)$. Therefore $((v-a_n)+K_\alpha,\ (v-a_n)+K_\alpha)+(\alpha(a_n{}^*(v-a_n))\longrightarrow 0$. It is easy that $\alpha(a_n{}^*(v-a_n))\longrightarrow 0$. Hence we have $v+K_\alpha=z$. Since $\alpha(a)=\alpha(v*a)$ for all $a\in A$, $A(1-v)\subset K_\alpha$. Therefore K_α is a proper modular left ideal of A. Let K be a maximal left ideal of A such that $K_\alpha\subset K$. Put $M=\{b+K_\alpha : b\in K\}$. Then M is a proper left modular ideal of A/K_α. Let cl(M) be a closure of M in $\mathscr{H}_\alpha$. We claim that cl(M) is a proper π_α-invariant subspace of $\mathscr{H}_\alpha$. It is sufficient to show that cl(M) is a proper subspace. For, let M_1 be a closure of M in $(A/K_\alpha, \|\cdot\|_q)$. Since $(A/K_\alpha, \|\cdot\|_q)$ is a Banach algebra, M_1 is proper. By the hypothesis, the identity mapping $(A/K_\alpha, |\cdot|_2)$ onto $(A/K_\alpha, \|\cdot\|_q)$ is continuous, so that M_1 is closed in $(A/K_\alpha, |\cdot|_2)$. Therefore there exists $x\in A/K_\alpha$ such that $x\notin M_1$. Let W be an open set in $\mathscr{H}_\alpha$ containing x such that $W\cap M_1=\phi$. M_2 denotes a closure of M_1 in $(\mathscr{H}_\alpha, |\cdot|_2)$. Then $W\cap M_2=\phi$. Therefore cl$(M)\cap W=\phi$. Hence cl(M) is proper. Since cl$(M)=(0)=K_\alpha$, $K=K_\alpha$. Hence K_α is a maximal left ideal of A.

PROPOSITION 3.2. *Let α be a pure state of A. Assume that there exists $K>0$ such that for all $a\in A$, $K\|a+K_\alpha\|_q\leq|a+K_\alpha|_2$. Suppose that $\{Tz : T\in\mathscr{B}\}$ is a closed* *set in* $\mathscr{H}_\alpha$ *for all nonzero* $z\in\mathscr{H}_\alpha$, *where* $\mathscr{B}$ *is the closure of* $\pi_\alpha(A)$ *in the operator norm. Then* $\mathscr{N}(\alpha)=\overline{K_\alpha+K_\alpha{}^*}$, *where* $\mathscr{N}(\alpha)$ *is the null space of* α.

PROOF. By Proposition 3.1, K_α is a modular left ideal of A. By the hypothesis, $\mathscr{B}$ acts strictly irreducible on $\mathscr{H}_\alpha$. By the similar method of Proposition 3.1 in [1], we obtain easily that $\mathscr{N}(\alpha)=\overline{K_\alpha+K_\alpha{}^*}$.

If α is a strictly pure state of A, π_α is a strictly irreducible *-representation. It follows that $\{Tz : T\in\mathscr{B}\} = \mathscr{H}_\alpha$. Hence we have

COROLLARY 3.3. (1, Proposition 3.1) *Let* α *be a strictly pure state of* A. *Then* $\mathscr{N}(\alpha)=\overline{K_\alpha+K_\alpha{}^*}$.

PROPOSITION 3.4. *Let* α *be as Proposition* 3.2. *Assume that* $\beta\in\mathscr{P}$, $M(\beta)=1$ *and* $K_\alpha=K_\beta$. *Then* $\alpha=\beta$.

PROOF. By the method of (1, Theorem 3.2), we have this proposition.

A strictly pure state α of A satisfies the hypothesis of Proposition 3.2. Hence we have

COROLLARY 3.5. (1, Theorem 3.2) *Let* α *be a strictly pure state of* A. *Assume that* $\beta\in\mathscr{P}$, $M(\beta)=1$ *and* $K_\alpha=K_\beta$. *Then* $\alpha=\beta$.

REMARK. Proposition 3.4 is a partial solution of Question 2 in §1.

4. Representations which are similar to π_α

Let $a\longrightarrow\pi(a)$ be a representation of A on $\mathscr{H}$. If there exists an isometric isomorphism of $\mathscr{H}$ onto $\mathscr{H}_\alpha$ and an invertible operator $V\in\mathscr{B}(H, H_\alpha)$ such that for all $a\in A$, $\pi(a)=V^{-1}\pi_\alpha(a)V$, then π is said to be *similar* to a strictly irreducible *-representation π_α. We consider the similarity of π_α in this section.

PROPOSITION 4.1. *Let* $a\longrightarrow\pi(a)$ *be a strictly irreducible representation of* A *on* $\mathscr{H}$. *Assume that* α *is a strictly pure state of* A, *with* $K_\alpha=K_\xi$, *where* $K_\xi=\{a\in A : \pi(a)\xi=0\}$ *for some nonzero* $\xi\in\mathscr{H}$. *Then* π *is similar to a strictly irreducible *-representation* π_α.

PROOF. Since π is strictly irreducible, $\{\pi(a)\xi : a\in A\}=\mathscr{H}$. We define a sesquilinear form $[\cdot,\cdot]$ on $\mathscr{H}\times\mathscr{H}$ by $[\pi(a)\xi, \pi(b)\xi]=\alpha(b^*a)$. Then there exists an invertible operator $U\in\mathscr{B}(\mathscr{H})$ such that $U=U^*$, $U\geq 0$ and $[\phi, \phi]=(U\phi, \phi)$, whenever $\phi, \phi\in\mathscr{H}$. Let $V\in\mathscr{B}(\mathscr{H})$ be a positive and invertible operator such that $V^2=U$. Then for all $\phi, \phi\in\mathscr{H}$, $[\phi, \phi]=(V\phi, V\phi)$. Let $\phi\in\mathscr{H}$. Then there exists $\phi\in\mathscr{H}$ such that $V\phi=\phi$, and exists $a\in A$ such that $\phi=\pi(a)\xi$. We define an operator T on $\mathscr{H}$ into $\mathscr{H}_\alpha$ by $T(\phi)=a+K_\alpha$. Since $K_\xi=K_\alpha$, T is well defined. Let ϕ be an element of $\mathscr{H}$. Then there exists $a\in A$ such that $\pi(a)\xi=\phi$. Therefore $\|V\phi\|^2=(V\phi, V\phi)=[\phi, \phi]=[\pi(a)\xi, \pi(a)\xi]=\alpha(a*a)=(|a+K_\alpha|_2)^2$ so that $\|V\phi\|=|a+K_\alpha|_2$. This proves that T is isometric. Clearly we have that T is isometric isomorphism. Let $W=TV$. Then $W\in\mathscr{B}(\mathscr{H}, \mathscr{H}_\alpha)$ and $W^{-1}\in\mathscr{B}(\mathscr{H}_\alpha, \mathscr{H})$. Since $W\pi(a)\pi(b)\xi=\pi_\alpha(a)W\pi(b)\xi$ for all $a, b\in A$, $\pi(a)=W^{-1}\pi_\alpha(a)W$. Hence π is similar to π_α.

Assume that A is a Banach *-algebra with the property that every modular maximal left ideal of A is the left kernel of a strictly pure state of A. By Proposition 4.1 and (2, Proposition 2.3 and 2.4), we have Corollary 4.2, 4.3 and 4.4.

COROLLARY 4.2. *Let $a\longrightarrow\pi(a)$ be as Proposition* 4.1. *Then π is similar to a strictly irreducible *-representation π_α, for some strictly pure state α.*

COROLLARY 4.3. *Let π be a continuous irreducible representation of A into $\mathscr{B}(X)$, where X denotes a Banach space. Assume that there exists $x\in X$ such that the left ideal $K_x=\{a\in A : \pi(a)x=0\}$ is modular maximal in A. Then π is similar to a strictly irreducible *-representation π_α, for some strictly pure state α.*

COROLLARY 4.4. *Let π be as Corollary* 4.3. *Assume that $A/ker(\pi)$ contains a minimal left ideal. Then the conclusion of Corollary* 4.3 *holds.*

REMARKS. Combining the above Proposition 4.1 and some results in [2,3], we have

(1) Let X be a reflexive Banach space. Assume that π is a continuous irreducible representation of a B^*-algebra A into $\mathscr{B}(X)$, and that there exists a nonempty subset S of A such that $W=\{\mathscr{N}(\pi(b)) : b\in S\}$ is a nonzero, finite dimensional subspace of X. Then π is similar to π_α, for some pure state α.

(2) Assume that π is continuous irreducible representation of a B^*-algebra A on $\mathscr{H}$. Let γ be as in the statement of (3, Theorem 3). Then γ is irreducible if and only if there exists a pure state α such that π is similar to π_α.

(3) The continuous irreducible representation π of a B^*-algebra on $\mathscr{H}$ is similar to π_α under the hypothesis of (3, Corollary 2), for some pure state α.

(4) Assume that A is a GCR-algebra (see [5]). Let π be a continuous irreducible representation of A on a Banach space X. Then π is similar to a irreducible *-representation π_α, for some pure state α.

(5) Let A be a B^*-algebra and let $A_n=\{a\in A: \|a\|\leq n\}$ for all positive integer n. Assume that π is a continuous irreducible representation of A into $\mathscr{B}(X)$, where X is a Banach space, and that there exists $x\in X$ such that

$$X=\bigcup_{n=1}^{+\infty} \text{the weak closue of the set } \pi(A_n)x \text{ in } X.$$

Then π is similar to π_α, for some pure state α.

Kyungpook University

REFERENCES

[1] B. A. Barnes, ***Strictly irreducible *-representations of Banach *-algebra,*** Trans. Amer. Math. Soc. 70 (1972), 459—469.

[2] ___________, ***Representation of B*-algebras on Banach spaces,*** Pacific Jour. of Math. 50 (1974), 7—18.

[3] ___________, ***The similarity problem for representations of a B*-algebra,*** Michigan Math. Jour. 22 (1975), 25—32.

[4] E. Hewitt and K. Ross, ***Abstract Harmornic Analysis,*** Vol. 1, Springer-Verlag, Berin, 1963.

[5] I. Kaplansky, ***The structure of certain operator algebras,*** Trans. Amer. Math. Soc. 70 (1951), 219—255.