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NUMERICAL PRINCIPAL AND ρ-PRINCIPAL POINTS OF AN OPERATOR

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1. Introduction

Let H be a complex Hilbert space. By an operator on H we mean a continuous linear transformation on H. As usual, let $\overline{W(T)}$ denote the closure of the numerical range $W(T)$ of an operator T. Define $H(T)$ as follows:

$$H(T)=\overline{W(T)}\cap\{\lambda:|\lambda|=\|T\|\},$$

the members of $H(T)$ are called principal points of an operator T. This concept originally due to Hildebrandt [2] was recently used by Shah and Sheth [7] to characterize normaloid operators. Let $P(T)$ denote the set $\sigma(T)\cap\{\lambda:|\lambda|=w(T)\}$ where $\sigma(T)$ is the spectrum of T and $w(T)$ is the numerical radius of an operator T on H. The purpose of this note is to characterize the spectraloid operators via this set $P(T)$; $P(T)$ will be called the set of numercal principal points of T.

In what follows $\gamma(T)$ denotes the spectral radius, $\sigma_p(T)$, the point spectrum, σ_{ap}(T), the approximate point spectrum of T; and the definitions of normaloid, convexoid and spectraloid operators are as usual.

In the last section we extend these considerations to ρ-oid operators the study of which was recently initiated by Patel and Gupta [6]. The discussion about it is postponed to that very section.

2. Some characterizations

We begin with the characterization of spectraloid operators.

THEOREM 2.1. *The set $P(T)$ is nonempty if and only if T is spectraloid.*

PROOF. Let $P(T)$ be nonempty. Then there exists λ such that $\lambda\in\sigma(T)$ and $|\lambda|=w(T)$. This means that $w(T)=|\lambda|\leq\gamma(T)$. Since $r(T)\leq w(T)$ always, T must be spectraloid.

Conversely, let T be spectraloid that is $r(T)=w(T)$. There exists λ in $\sigma(T)$ such that $r(T)=|\lambda|$. Hence $\lambda\in\sigma(T)$ and $|\lambda|=w(T)$; thus we have $\lambda\in P(T)$ and we are through.

The following result shows under what conditions on operator T, one has $P(T)=\sigma(T)$.

THEOREM 2.2. *If T is unitary or a scalar multiple of a unitary operator, then $P(T)=\sigma(T)$.*

PROOF. If $T=zU$, U being unitary, z being scalar, then

$$\begin{aligned} P(T)=P(zU) &= \sigma(zU)\cap\{\lambda : |\lambda|=w(zU)\} \\ &= \sigma(zU)\cap\{\lambda : |\lambda|=|z|\}=\sigma(zU) \\ &= \sigma(T). \end{aligned}$$

It is also observed that the hypothesis of the preceding theorem implies that $P(T^{-1})=[P(T)]^{-1}$. As the inverse of an invertible spectraloid operator need not be spectraloid, it may well happen that $P(T^{-1})$ may not equal $[P(T)]^{-1}$. Hence let us ask:

i) If $P(T)=\sigma(T)$, does it follow that T is a scalar multiple of a unitary operator?

ii) If $P(T^{-1})=[P(T)]^{-1}$, where T is invertible spectraloid, does it follow that T is a scalar multiple of a unitary operator?

We answer these questions under rather restrictive assumptions in the following:

THEOREM 2.3. *Let T be an operator on H such that $P(T)=\sigma(T)$ lies on the unit circle. Then T is unitary if* i) $P(T^{-1})=[P(T)]^{-1}$ *or* ii) T *satisfies the growth condition G_1 i.e. $\|(T-\lambda I)^{-1}\|\leq[d(\lambda,\sigma(T))]^{-1}$ for all $\lambda\notin\sigma(T)$.*

PROOF. (i) Clearly $P(T)$ is nonempty and thus T is spectraloid. Also $P(T^{-1}) = [(P(T)]^{-1}$; which implies that T^1 is spectraloid. Now $1=r(T)=w(T)$ and $1=r(T^{-1})=w(T^{-1})$. Thus $W(T^{\pm1})\subset\triangle$, the unit disc of the complex plane. By applying the result due to Stampfli [8] we arrive at the desired conclusion.

(ii) Clearly $P(T)$ is nonempty and hence T is spectraloid. Now $r(T)=1$ implies that $w(T)=1$. Since T satisfies the growth condition G_1, $\|T^{-1}\|\leq 1$. Hence $w(T^{-1})\leq 1$. Again by applying the result due to Stampfli [8] we are through.

3. Additional results

It is noted that if T is a scalar multiple of the identity then $P(T+\lambda I)=P(T)+\lambda$, λ scalar. Now let us ask: What one can say about the converse? In order to answer this we need the following two lemmas.

LEMMA 3.1, *If T is convexoid operator with $\sigma(T)$ a singleton set, then T is a scalar multiple of the identity.*

PROOF. If $\sigma(T)=\{\mu\}$, then $W(T)=\{\mu\}$ and we are through.

LEMMA 3.2. *For a spectraloid operator T, if $P(T+\lambda I)=P(T)+\lambda$ for every scalar λ, then T is convexoid.*

PROOF. The hypothesis implies that $P(T+\lambda I)$ is nonempty for every scalar λ. Hence $T+\lambda I$ is spectraloid which in view of the result due to Furuta [1] implies that T is convexoid.

We now prove the main result of this section.

THEOREM 3.3. *For a spectraloid operator T on H if $P(T+\lambda I)=P(T)+\lambda$ for every scalar λ, then T is a scalar multiple of the identity.*

PROOF. In view of Lemmas 3.1 and 3.2, it suffices to show that $\sigma(T)$ is singleton. Firstly we prove that $P(T)$ is singleton set. That $P(T)$ is nonempty is obvious as T is spectraloid. Let λ_1, $\lambda_2 \in P(T)$ with $\lambda_1 \neq \lambda_2$. Then for any scalar λ, $\lambda_1+\lambda$ and $\lambda_2+\lambda$ are in $P(T)+\lambda=P(T+\lambda I)$. This implies that

$$|\lambda_1+\lambda|=w(T+\lambda I)=|\lambda_2+\lambda| \tag{1}$$

If scalar λ_0 is not on the perpendicular bisector of the line segment joining λ_1 and λ_2, then $|\lambda_1-\lambda_0| \neq |\lambda_2-\lambda_0|$ which contradicts (1) for $\lambda=-\lambda_0$. Hence $P(T)$ must be singleton set. Let $P(T)=\{\mu_0\}$. Now we shall prove that $\sigma(T)=\{\mu_0\}$. If possible, let $\mu \neq \mu_0 \in \sigma(T)$. Since $\mu \notin P(T)$, $|\mu|<w(T)$. Select a scalar λ such that

$$|\mu-\lambda|>|\mu_0-\lambda| \tag{2}$$

Now $\mu_0-\lambda \in P(T)-\lambda=P(T-\lambda I)$ which implies that $|\mu_0-\lambda|=w(T-\lambda I)$. Since $\mu-\lambda \in \sigma(T-\lambda I)$, it follows that $|\mu-\lambda I| \leq w(T-\lambda I)$. Hence $|\mu-\lambda| \leq |\mu_0-\lambda|$ which is a contradiction to (2). Hence $\sigma(T)$ must be singleton set and in fact $\sigma(T)=\{\mu_0\}$.

4. ρ-Principal points of an operator

Let $C_\rho (\rho>0)$ be the class of all operators with unitary ρ-dilation in the sense of [4] ; let $R(T)=\sigma(T) \cap \{\lambda : |\lambda|=w_\rho(T)\}$, where $w_\rho(T)$ is the operator radius of T defined as

$$w_\rho(T)=\inf\{\alpha : \alpha>0,\ \alpha^{-1}T \in C_\rho\}$$

(See Holbrook [3], Patel [5], Patel-Gupta [6]). The members of $R(T)$, will be called ρ-principal points of an operator T.

An operator T is called ρ-oid if $w_\rho(T)=r(T)$. On the lines of our previous discussion we obtain.

THEOREM 4.1. *The set $R(T)$ is nonempty if and only if T is ρ-oid*

PROOF. Let T be ρ-oid. Then $w_\rho(T)=r(T)$ and hence for some $\lambda\in\sigma(T)$, $|\lambda| =r(T)=w_\rho(T)$. This means that $\lambda\in R(T)$ and thus $R(T)$ is nonempty. Conversely let $R(T)$ be nonempty. Take $\lambda\in R(T)$. Then $\lambda\in\sigma(T)$ with $|\lambda|=w_\rho(T)$. Hence $w_\rho(T)=|\lambda|\leq r(T)=\lim_{\alpha\to\infty} w_\alpha(T)\leq w_\rho(T)$. Hence $w_\rho(T)=r(T)$ which means that T is ρ-oid.

We also have an analogue of theorem 2.3. For that we need to recall the following concept. T is called an operator of class $M_\rho(\rho\geq1)$ if $w_\rho[(T-zI)^{-1}] =\frac{1}{d(z,\sigma(T))}$; equivalently if $(T-zI)^{-1}$ is ρ-oid for all $z\notin\sigma(T)$.

THEOREM 4.2. *Let T be an operator on H such that $R(T)=\sigma(T)$ (Assume $\rho\geq1$) lies on the unit circle. Then T is unitary if* i) $R(T^{-1})=(R(T))^{-1}$, *or* (ii) *T is an operator of class M_δ, $\delta\geq1$.*

PROOF. Clearly $R(T)$ is nonempty and hence T is ρ-oid. (i) $R(T^{-1})=[R(T)]^{-1}$ implies that T^{-1} is ρ-oid. Now $w_\rho(T)=r(T)=1$ and $w_\rho(T^{-1})=r(T^{-1})=1$ which implies that $T\in C_\rho$ and $T^{-1}\in C_\rho$. Hence T is unitary in view of the Corollary 4 of [9].

(ii) As $w_\rho(T)=r(T)=1$ we have that $T\in C_\rho$. Also since $T\in M_\delta$, $(T-zI)^{-1}$is δ-oid for all $z\notin\sigma(T)$. In particular, for $z=0$, T^{-1} is δ-oid. Hence $w_\delta(T^{-1}) =r(T^{-1})=1$ imply that $T^{-1}\in C_\delta$. Hence T is unitary in view of the Corollary 4 of [9].

Here is an analogue of the result stated in Section 3, the proof of which is straight forward.

THEOREM 4.3. *If T is a scalar multiple of the identity then $R(T+\lambda I)=R(T)+\lambda$, λ scalar.*

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REFERENCES

[1] Furuta T. and Nakamoto R., *On the numerical range of an operator*, Proc. Japan Acad. 47 (3), 1971, 279-284.

[2] Hilderbrandt S., *The closure of the numerical range of an operator as a spectral set*, Comm. Pure Appl. Math. 17 (1964) 415-421.

[3] Holbrook J.A.R., *On the power bounded operators of Sz.-Nagy and Foias*, Acta sci. Math. (Szeged) 29 (1968), 299-310, MR 39#810.

[4] B.Sz. Nagy and C.Foias, *Analyse harmonique des operateurs de l'espace de Hilbert*, Masson, Paris, Akad. Kiado, Budapest, 1967, English rev. transl., North Holland, Amsterdam, American Elsevier, New York, Akad. Kiado, Budapest, 1970, MR 37 # 778; 43 # 947.

[5] Patel S.M., *On generalized numerical ranges*, to appear in Paci. J. Math. 64, .1, 1976.

[6] Patel S.M. and Gupta B.C., *Operators satisfying certain growth conditions*, Proc. AMS. 53, 2, 1975, 341-346.

[7] Shah N.C. and Sheth I. H., *The principal points of an operator* (preprint) (to appear).

[8] Stampfli J.G., *Minimal range theorems for operators with thin spectra*, Paci. Jour. Math. 23(3), 1967, 601—612.

[9] Stampfli J.G., *A local spectral theory for operators*, Jour. Fun. Ana. 4, (1), 1969, 1—10.