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SUMMATION FORMULAE FOR ${}_2F_1\left(\frac{1}{2}\right)$

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0. Abstract

In this paper we prove three summation formulae for hypergeometric series ${}_2F_1\left(\frac{1}{2}\right)$.

1. Introduction

Gauss [1, p. 243(III.6)] and Bailey [1, p. 243(III.7)] have given only two summation formulae for hypergeometric series ${}_2F_1\left(\frac{1}{2}\right)$. The summation formulae are useful in the solution of certain problems in theoretical physics and optimization problems in Management Sciences. The object of this paper is to prove three summation formulae for ${}_2F_1\left(\frac{1}{2}\right)$ and later on use these results to obtain summation formulae for ${}_2F_1(-1)$. In the investigation we use the results due to Bailey [1, p. 245, (III.20)]

$$(1)\qquad {}_4F_3\left[\begin{matrix}\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha,\ \beta+n,\ -n:1\\ \frac{1}{2}\beta,\ \frac{1}{2}+\frac{1}{2}\beta,\ 1+\alpha;\end{matrix}\right]=\frac{(\beta-\alpha)_n}{(\beta)_n},$$

$$(2)\qquad {}_2F_1\left[\begin{matrix}\alpha, 1-\alpha:\frac{1}{2}\\ \beta\quad ;\end{matrix}\right]=\frac{\Gamma\left(\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)}{\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{2}\alpha\right)},$$

Gauss [1, p. 243(III.6)]

$$(3)\qquad {}_2F_1\left[\begin{matrix}\alpha,\beta:1\\ \gamma\quad ;\end{matrix}\right]=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

valid for $R(\gamma-\alpha-\beta)>0$ and Watson [1, p. 245, (III.23)]

$$(4)\qquad {}_3F_2\left[\alpha,\beta,\gamma:\frac{1}{2}(1+\alpha+\beta),2\gamma:1\right]$$
$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\gamma\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2}\gamma\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\gamma\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\beta+\gamma\right)},$$

valid for $R(2\gamma-\alpha-\beta)>-1$.

2. The first summation formula to be proved is

$$(5)\quad {}_2F_1\left[\begin{matrix}\alpha-\beta, 1+\beta\,;\,\frac{1}{2}\\ \frac{1}{2}\alpha\,;\end{matrix}\right]=\frac{2^{1+\frac{1}{2}\alpha}\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\beta\right)}{\Gamma(1+\beta)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)\Gamma\left(\frac{3}{4}\alpha-\frac{1}{2}\beta\right)},$$

valid for $R\left(\frac{1}{2}-\frac{1}{2}\alpha+\beta\right)>0.$

PROOF. To prove (5), we start with the left side of (5)

$$ {}_2F_1\left[\alpha-\beta, 1+\beta\,;\,\frac{1}{2}\alpha\,;\,\frac{1}{2}\right]=\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_n(1-\beta)_n}{\left(\frac{1}{2}\alpha\right)_n n!\,2^n}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_n}{\left(\frac{1}{2}\alpha\right)_n n!}2^n\sum_{r=0}^{n}\frac{(-n)_r(\alpha-\beta)_r\left(\frac{1}{2}\alpha\right)_r\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_r(1+\beta-\alpha)_n}{(1+\alpha)_r\left(\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}n\right)_r\left(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}n\right)_r r!}$$

by (1)

$$=\sum_{r=0}^{\infty}\frac{(\alpha-\beta)_r\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_r 2^r}{(1+\alpha)_r r!}\,{}_2F_1\left[\begin{matrix}\alpha-\beta+r, 1+\beta-\alpha-r\,;\,\frac{1}{2}\\ \frac{1}{2}\alpha+r\end{matrix}\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\alpha\right)\sqrt{\pi}}{2^{2\alpha-1}\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)\Gamma\left(\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}\times$$

$$ {}_2F_1\left[\frac{1}{2}+\frac{1}{2}\alpha, \alpha-\beta\,;\,1+\alpha\,;\,1\right]$$

by (2)

$$=\frac{2^{1+\frac{1}{2}\alpha}\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\beta\right)}{\Gamma(1+\beta)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)\Gamma\left(\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}$$

by (3)

This completes the proof of (5).

In case $\beta=0$ in (5), it gives

$$(6)\quad {}_2F_1\left[\begin{matrix}\alpha,\ 1\,;\,\frac{1}{2}\\ \frac{1}{2}\alpha\end{matrix}\right]=\frac{2^{1+\frac{1}{2}\alpha}\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{4}\alpha\right)\Gamma\left(\frac{3}{4}\alpha\right)}$$

valid for $0<R(\alpha)\leqslant 1$

3. The second summation formula to be proved is

$$(7)\quad {}_2F_1\left[\begin{matrix}\alpha-\beta,\ 1+\beta\,;\,\frac{1}{2}\\ \frac{1}{2}+\frac{1}{2}\alpha\,;\end{matrix}\right]=\frac{2^{\frac{1}{2}(\alpha+1)}\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\beta-\frac{1}{2}\alpha\right)}{\Gamma(1+\beta)\Gamma\left(\frac{1}{4}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)},$$

valid for $R\left(1+\beta-\frac{1}{2}\alpha\right))>0.$

PROOF. It can be proved in the same way as (5).
In particular, if $\beta=0$ in (7), it gives

$$(8)\quad {}_2F_1\left[\begin{matrix}\alpha,\ 1\ ;\ 1\\ \frac{1}{2}+\frac{1}{2}\alpha\end{matrix}\right]=\frac{2^{\frac{1}{2}(\alpha+1)}\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1-\frac{1}{2}\alpha\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\alpha\right)\Gamma\left(\frac{3}{4}-\frac{1}{4}\alpha\right)},$$

valid for $-1<R(\alpha)<2$.

4. The third summation formula to be proved is

$$(9)\quad {}_2F_1\left[\begin{matrix}\alpha-\beta,\ 1+\beta\ ;\ \frac{1}{2}\\ \frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\ ;\end{matrix}\right]$$

$$=\frac{2^{\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{4}\alpha}\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}{\Gamma\left(1+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{4}+\frac{7}{8}\alpha-\frac{3}{4}\beta\right)\Gamma\left(\frac{3}{4}-\frac{1}{8}\alpha+\frac{1}{4}\beta\right)}$$

valid for $R\left(1+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)>0$

PROOF. To prove (9), we start with the left side of (9)

$${}_2F_1\left[\begin{matrix}\alpha-\beta, 1+\beta\ ;\ \frac{1}{2}\\ \frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\ ;\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_n(1+\beta)_n}{\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)_n n!2^n}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_n}{\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)_n n!2^n}$$

$$\times\sum_{r=0}^{\infty}\frac{(-n)_r(\alpha-\beta)_r\left(\frac{1}{2}\alpha\right)_r}{(1+\alpha)_r\left(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta-\frac{1}{2}n\right)_r}\ \frac{\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_r(1+\beta-\alpha)_n}{\left(\frac{1}{2}-\frac{1}{2}\beta-\frac{1}{2}n\right)_r r!}\qquad\text{by (1)}$$

$$=\sum_{r=0}^{\infty}\frac{(\alpha-\beta)_r\left(\frac{1}{2}\alpha\right)_r\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_r 2^r}{(1+\alpha)_r\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)_r r!}$$

$${}_2F_1\left[\begin{matrix}\alpha-\beta+r, 1+\beta-\alpha-r\ ;\ \frac{1}{2}\\ \frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta+r\ ;\end{matrix}\right]$$

$$=\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}{2^{\frac{3}{4}\alpha-\frac{1}{2}\beta-\frac{1}{2}}\Gamma\left(\frac{1}{4}+\frac{7}{8}\alpha-\frac{3}{4}\beta\right)\Gamma\left(\frac{3}{4}-\frac{1}{4}\alpha+\frac{1}{4}\beta\right)}$$

$$
{}_3F_2\left[\begin{matrix}\frac{1}{2}\alpha, \alpha-\beta, \frac{1}{2}+\frac{1}{2}\alpha : 1\\ \frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta, 1+\alpha :\end{matrix}\right] \qquad \text{by (2)}
$$

$$
=\frac{2^{\frac{1}{2}\left(1+\beta-\frac{1}{2}\alpha\right)}\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}{\Gamma\left(1+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{4}+\frac{7}{8}\alpha-\frac{3}{4}\beta\right)}
$$

$$
\times\frac{\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}{\Gamma\left(\frac{3}{4}-\frac{1}{8}\alpha+\frac{1}{4}\beta\right)} \qquad \text{by (4).}
$$

This completes the proof of (9) under the conditions stated with (9).

In case $\beta=0$ in (9), it gives

(10) ${}_2F_1\left[\alpha, 1 : \frac{1}{2}+\frac{3}{4}\alpha : \frac{1}{2}\right]$

$$
=\frac{2^{\frac{1}{2}-\frac{1}{4}\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{4}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha\right)\Gamma\left(\frac{1}{4}+\frac{1}{4}\alpha\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{4}+\frac{7}{8}\alpha\right)\Gamma\left(\frac{3}{4}-\frac{1}{8}\alpha\right)}
$$

valid for $-\frac{2}{3}<R(\alpha)<4$.

5. Applications

In this section we apply the formulae (5), (7) and (9) to obtain summation formulae for ${}_2F_1(-1)$. In the investigation we require the formula [1, p. 31 (1.7.14)].

(11) ${}_2F_1\left[a, c-b : c : -1\right]=2^{-a}\ {}_2F_1\left[a, b : c : \frac{1}{2}\right]$, valid for $R(b-a)>-1$.

We use (5) in (11), we get

(12) ${}_2F_1\left[\alpha-\beta, \frac{1}{2}\alpha-1-\beta : \frac{1}{2}\alpha : -1\right]$

$$
=\frac{2^{\frac{1}{2}+\beta-\frac{1}{2}\alpha}\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\beta-\frac{1}{2}\alpha\right)}{\Gamma(1+\beta)\Gamma\left(\frac{1}{4}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)},
$$

valid for $R\left(1+\beta-\frac{1}{2}\alpha\right)>0$.

Finally we use (9) in (11), it yields

(14) ${}_2F_1\left[\alpha-\beta, \frac{3}{4}\alpha-\frac{3}{2}\beta-\frac{1}{2} : \frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta : -1\right]$

$$=\frac{2^{\frac{1}{2}+\frac{3}{2}\beta-\frac{5}{4}\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\beta-\frac{1}{4}\alpha\right)\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{3}{4}\alpha-\frac{1}{2}\beta\right)}{\Gamma\left(1+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{4}+\frac{7}{8}\alpha-\frac{3}{4}\beta\right)\Gamma\left(\frac{3}{4}-\frac{1}{8}\alpha+\frac{1}{4}\beta\right)}$$

valid for $R\left(\frac{1}{2}+\frac{1}{2}\beta-\frac{1}{4}\right)>0.$

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