Kyungpook Math. J.
Volume 18, Number 2
December, 1978

DOUBLE FINITE SUMMATION FORMULAS FOR THE H-FUNCTION OF TWO VARIABLES

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1. Introduction

The parameters of the H-function of two variables [3, p.117] occurring in the present paper will be displayed in the following contracted notation (analogous to the notation of Srivastava and Panda [6]):

$$H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}((a_{p_1}:\alpha_{p_1},A_{p_1})):((c_{p_2},\varepsilon_{p_2})):((e_{p_3},E_{p_3}))\\((b_{q_1}:\beta_{q_1},B_{q_1})):((d_{q_2},\delta_{q_2})):((f_{q_3},F_{q_3}))\end{matrix}\right]$$

$$=(2\pi i)^{-2}\int_{L_1}\int_{L_2}\phi(u,v)\ \theta_1(u)\ \theta_2(v)\ x^u\ y^v\ du\ dv \tag{1.1}$$

where, for convenience, $((a_{p_1}:\alpha_{p_1},A_{p_1}))$ abbreviate the p_1-parameter sequence $(a_1:\alpha_1,A_1),\cdots,(a_{p_1}:\alpha_{p_1},A_{p_1}):((c_{p_2},\varepsilon_{p_2}))$ for p_2-parameter sequence $(c_1,\varepsilon_1),\cdots,(c_{p_2},\varepsilon_{p_2})$ and similar interpretations for $((b_{q_1}:\beta_{q_1},B_{q_1})),((d_{q_2},\delta_{q_2}))$ etc.

Also $$\phi(u,v)=\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_ju+A_jv)\left[\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_ju-A_jv)\times\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_ju+B_jv)\right]^{-1}$$

$$\theta_1(u)=\prod_{j=1}^{n_2}\Gamma(1-c_j+\varepsilon_ju)\prod_{j=1}^{m_2}\Gamma(d_j-\delta_ju)\left[\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\varepsilon_ju)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_ju)\right]^{-1}$$

and with $\theta_2(v)$ defined analogously in terms of the parameter sets $((e_{p_3},E_{p_3}))$, $((f_{q_3},F_{q_3}))$. An empty product is interpreted as unity, and all the greek and capital letters are positive. The nature of contours L_1 and L_2 in (1.1), the conditions on parameters of this function, its asymptotic expansions, particular cases etc., can be found in a recent paper by Goyal [3].

The aim of this paper is to establish three interesting double finite summation formulas for the H-function of two variables. By suitably specializing the various parameters involved, these formulas would yield the corresponding double summation formulas for a large variety of special functions involving one or two variables. The corresponding formulas for Kampé de Fériet function, Fox's H-function, generalized hypergeometric function have been obtained in this paper. In the end of the paper, a double summation formula has been reduced to single summation formula involving the H-function of two variables.

To save space, three dots, $\cdots$, appearing at a particular place in any H-function of two variables will display that the parameters in that position are the same as that of the H-function of two variables defined by (1.1).

2. Summation formulas

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b-e-n+1)_r(b)_s}{r!s!(2-e-m-n)_r(e)_s}$$
$$\times H^{0\;\;\;,n_1+1:\cdots:\cdots}_{p_1+1,\,q_1+2:\cdots:\cdots}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-c-r-s:\alpha,\beta),\cdots & :\cdots;\cdots\\ \cdots,(1-c-r:\alpha,\beta),(1-c-s:\alpha,\beta) & :\cdots;\cdots\end{matrix}\right]$$
$$=\frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}$$
$$\times H^{0\;\;\;,n_1+1:\cdots:\cdots}_{p+1,\,q+2:\cdots:\cdots}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-c-m-n:\alpha,\beta),\cdots & :\cdots;\cdots\\ \cdots,(1-c-m:\alpha,\beta),(1-c-n:\alpha,\beta) & :\cdots;\cdots\end{matrix}\right] \tag{2.1}$$

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b)_s}{r!\;\;s!}$$
$$\times H^{A}_{B}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-c-r-s:\alpha,\beta),\cdots & :\cdots,(2-e-m-n+r,\varepsilon) & ;\cdots\\ \cdots,(1-c-r:\alpha,\beta),(1-c-s:\alpha,\beta) & :(1-e-n+b+r,\varepsilon),\cdots,(1-e-s,\varepsilon) & ;\cdots\end{matrix}\right]$$
$$=(-1)^{m+n}(b)_m$$
$$\times H^{A}_{B}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-c-m-n:\alpha,\beta),\cdots & :\cdots,(2-e-n,\varepsilon) & ;\cdots\\ \cdots,(1-c-m:\alpha,\beta),(1-c-n:\alpha,\beta) & :(1-e+b,\varepsilon),\cdots,(1-e-n,\varepsilon) & ;\cdots\end{matrix}\right] \tag{2.2}$$

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(a)_r}{r!\;\;s!}$$
$$\times H^{A}_{B}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-c-r-s;\alpha,\beta),\cdots & :\cdots,(2-m-n-d+s,\varepsilon) & ;\cdots\\ \cdots,(1-c-r;\alpha,\beta),(1-c-s:\alpha,\beta) & :(1+a-d-m+s,\varepsilon),\cdots,(1-d-r,\varepsilon) & ;\cdots\end{matrix}\right]$$
$$=(-1)^{m+n}(a)_n$$
$$\times H^{A}_{B}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-c-m-n:\alpha,\beta),\cdots & :\cdots,(2-d-m,\varepsilon) & ;\cdots\\ \cdots,(1-c-m:\alpha,\beta),(1-c-n:\alpha,\beta) & :(1+a-d,\varepsilon),\cdots,(1-d-m,\varepsilon) & ;\cdots\end{matrix}\right] \tag{2.3}$$

In (2.2) and (2.3) above, A stands for $0\;\;\;,n_1+1:m_2+1,n_2;m_3,n_3$ and B for $p_1+1,q_1+2:p_2+1,q_2+2;p_3,q_3$.

Conditions corresponding appropriately to the conditions (i) to (vi) on p. 119 in the paper by Goyal [3] are assumed to be satisfied by all the H-function of two variables occurring in the above formulas.

PROOF of (2.1) : Writing Mellin-Barnes integral of the H-function of two variables from (1.1) in the left hand side of (2.1), changing the order of integration and summation, which is justified as the series involved are finite, we find that

$$\text{l. h. s.} = (2\pi i)^{-2}\int_{L_1}\int_{L_2}\frac{\phi(u,v)\theta_1(u)\theta_2(v)x^u y^v}{\Gamma(c+\alpha u+\beta v)}$$
$$\times\left[\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b-e-n+1)_r(b)_s(c+\alpha u+\beta v)_{r+s}}{r!\,s!\,(2-e-m-n)_r(e)_s(c+\alpha u+\beta v)_r(c+\alpha u+\beta v)_s}\right]du\,dv$$

Using the following result due to Carlitz [2, p. 139] :

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(a)_r(b)_s(c)_{r+s}}{r!\,s!\,(d)_r(e)_s(c)_r(c)_s}=\frac{(b)_m(a)_n(c)_{m+n}}{(b-a)_m(a-b)_n(c)_m(c)_n} \tag{2.4}$$

($a-b$ is not an integer, $a=b-e-n+1$, and $b=a-d-m+1$)
in the above expression and interpreting the result thus obtained with the help of (1.1), we get the right hand side of (2.1).

The results (2.2) and (2.3) can be easily proved by proceeding on the lines similar to the result (2.1).

3. Special cases of (2.1)

(a) On taking $m_2=m_3=1$, $n_1=p_1$, $n_2=n_3=p_2=p_3$, $q_2=q_3$, $d_1=f_1=0$, replacing q_2 by q_2+1, putting all Greek and capital letters equal to one in (2.1), and using a recent result due to Goyal [3, p. 123, (3.3)] therein, we get the following double summation formula for the Kampé de Fériet function [1]:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b-e-n+1)_r(b)_s(c)_{r+s}}{r!\,s!\,(2-e-m-n)_r(e)_s(c)_r(c)_s}$$
$$\times F^{p_1+1,p_2}_{q_1+2,q_2}\left[\begin{matrix}c+r+s,(a_{p_1}):(c_{p_2});(e_{p_2})\\ c+r,c+s,(b_{q_1}):(d_{q_2});(f_{q_2})\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right]$$
$$=\frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}F^{p_1+1,p_2}_{q_1+2,q_2}\left[\begin{matrix}c+m+n, & (a_{p_1}):(c_{p_2});(e_{p_2})\\ c+m, & c+n,(b_{q_1}):(d_{q_2});(f_{q_2})\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right] \tag{3.1}$$

(b) Again, on putting $n_1=p_1=q_1=n_3=p_3=0$, $m_3=q_3=1$, $f_1=0$, $F_1=1$, $\beta=1$ in (2.1), letting $y\to 0$, and using [3, p. 123, (3.5)] therein, we get the following series for Fox's H-function, which is believed to be new:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b-e-n+1)_r(b)_s}{r!\,s!\,(2-e-m-n)_r(e)_s}$$

$$\times H^{m_2,n_2+1}_{p_2+1,q_2+2}\left[x\left|\begin{matrix}(1-c-r-s,\alpha),((c_{p_2},\varepsilon_{p_2}))\\((d_{q_2},\delta_{q_2})),(1-c-r,\alpha),(1-c-s,\alpha)\end{matrix}\right.\right]$$

$$=\frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}H^{m_2,n_2+1}_{p_2+1,q_2+2}\left[x\left|\begin{matrix}(1-c-m-n,\alpha),((c_{p_2},\varepsilon_{p_2}))\\((d_{q_2},\delta_{q_2})),(1-c-m,\alpha),(1-c-n,\alpha)\end{matrix}\right.\right] \tag{3.2}$$

(c) Further, on taking $m_2=1$, $n_2=p_2$, $d_1=0$, all Greek letters equal to one in (3.2) and using the result [4, p.600, (4.6)], we get

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b-e-n+1)_r(b)_s(c)_{r+s}}{r!\,s!\,(2-e-m-n)_r(e)_s(c)_r(c)_s}\,{}_{p+1}F_{q+2}\left[\begin{matrix}c+r+s,(c_p)\\(d_q),c+r,c+s\end{matrix};x\right]$$

$$=\frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}\,{}_{p+1}F_{q+2}\left[\begin{matrix}c+m+n,(c_p)\\(d_q),c+m,c+n\end{matrix};x\right] \tag{3.3}$$

Similar type of double summation formulas for (2.2) and (2.3) can be obtained by specializing the parameters of the H-function of two variables as indicated above, but we shall not record them here due to the triviality of the analysis.

Lastly, if we put $n=b=0$, $1-e=a$ in (2.2), the double series reduces to the following single series for the H-function of two variables.

$$\sum_{r=0}^{m}\frac{(-m)_r}{r!}H^{\cdots:m_2+1,n_2:\cdots}_{\cdots:p_2+1,q_2+1:\cdots}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}\cdots:\cdots,(a-m+r+1,\varepsilon):\cdots\\\cdots:(a,\varepsilon),\cdots\quad;\cdots\end{matrix}\right.\right]$$

$$=(-1)^m H^{\cdots:m_2+1,n_2:\cdots}_{\cdots:p_2+1,q_2+1:\cdots}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}\cdots:\cdots,(a+1,\varepsilon):\cdots\\\cdots:(a,\varepsilon),\cdots\quad;\cdots\end{matrix}\right.\right] \tag{3.4}$$

If we reduce the H-function of two variables to Fox's H-function with the help of a known result due to Goyal [3, p.123, (3.5)], we get the single series for the H-function, which is a particular case of a general result given earlier by Jain [5, p.462].

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