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ON INTEGRAL MODULUS OF CONTINUITY OF FOURIER SERIES

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1. Let F be a function with period 2π. If $F\in L_p(1\leq p<\infty)$, then the L_p-modulus of continuity of order $k\geq 1$ of F is defined by

$$\omega_b^k(\delta\,;F)=\sup_{0<|t|\leq\delta}\|\Delta_t^k F(x)\|_{L_p},$$

where

$$\Delta_t^k F(x)=\sum_{\alpha=0}^{k}(-1)^{k-\alpha}\binom{k}{\alpha}F(x+\alpha t)$$

and $\|\cdot\|$ denotes the norm.

A sequence $\{a_v\}$ is said to be ***quasi-convex*** if $\sum_{v=1}^{\infty}v|\Delta^2 a_{v-1}|<\infty$,

where

$$\Delta^2 a_{v-1}=\Delta a_{v-1}-\Delta a_v=a_{v-1}-2a_v+a_{v+1}.$$

Throughout this paper the letter C_k denotes an absolute constant depending on k which may have different values in different contexts.

2. Concerning the series $g(x)=\sum_{v=1}^{\infty}a_v\sin vx$, Teljakovskiĭ [3] has proven the following

THEOREM A. *Let $\{a_v\}$ be a quasi-convex null sequence and $\sum_{v=1}^{\infty}v^{-1}|a_v|<\infty$. Then $g(x)$ is integrable and its integral modulus of continuity satisfies the condition*

$$\omega(1/n\,;g)=2/\pi\sum_{v=n}^{\infty}v^{-1}|a_v|+O(A_n),$$

where

$$A_n=n^{-1}\sum_{v=1}^{n}v^2|\Delta^2 a_{v-1}|+\sum_{v=n+1}^{\infty}v|\Delta^2 a_{v-1}|.$$

The object of this paper is to generalize Theorem A for modulus of continuity of order k by proving the following.

THEOREM. *Let* $\{a_v\}$ *be a quasi-convex null sequence and* $\sum_{v=1}^{\infty} v^{-1}|a_v|<\infty$. *Then*

$$\omega^k(1/n : g)=C_k\Big[\sum_{v=n}^{\infty} v^{-1}|a_v|+A_n+B_n\Big],$$

where

$$B_n=n^{-k}\sum_{v=1}^{n} v^{k+1}|\Delta^2 a_{v-1}|+\sum_{v=n+1}^{\infty} v|\Delta^2 a_{v-1}|.$$

The following theorem of Aljančic [1] is a corollary of our theorem.

THEOREM B. *Let* $\{a_v\}$ *be a convex null sequence and* $\sum_{v=1}^{\infty} v^{-1} a_v<\infty$.

Then

$$\omega^k(1/n : g)\leq C_k n^{-k}\sum_{v=1}^{n} v^{k-1} a_v+\sum_{v=n}^{\infty} v^{-1} a_v.$$

3. We require the following lemma for the proof of our theorem.

LEMMA 1. [1]. *Let* $0<t\leq n^{-1}$ $(n=1,2,\cdots)$ *and* k *a natural number. If* $\bar{K}_v(x)$ *denotes the Fejer conjugate kernel, then*

$$\int_0^{\pi}|\Delta_{\pm t}^k \bar{K}_v(x)|dx\leq C_k v^k t^k \quad (v=1,2,\cdots,n),$$

$$\int_{(k+1)/n}^{\pi} |\Delta_{\pm t}^k \bar{K}_v(x)|dx\leq C_k (v=1,2,\cdots).$$

4. PROOF OF THE THEOREM. The function g is integrable under the given hypothesis. To prove the theorem it suffices to show that for $0<t\leq n^{-1}$

$$\int_0^{\pi}|\Delta_{\pm t}^k g(x)|dx\leq C_k\Big[\sum_{v=n}^{\infty} v^{-1}|a_v|+A_n+B_n\Big].$$

we write

$$(4.1)\quad \int_0^{\pi}|\Delta_{\pm t}^k g(x)|dx=\int_0^{(k+1)/n}+\int_{(k+1)/n}^{\pi}$$
$$=I_1+I_2.$$

We shall estimate I_1 and I_2 separately. Let

$$\overline{D}_0(x) = -1/2 \cot x/2$$

$$\overline{D}_v(x) = \overline{D}_0(x) + \sin x + \sin 2x + \cdots + \sin vx$$

$$= -\frac{\cos \frac{2v+1}{2} x}{2 \sin x/2}, \quad v = 1, 2, \cdots.$$

$$\overline{F}_v(x) = \overline{D}_0(x) + \overline{D}_1(x) + \cdots + \overline{D}_v(x)$$

$$= -\frac{\sin(v+1)x}{4 \sin^2 x/2}, \quad v = 0, 1, 2, \cdots.$$

By twice use of Abel's transformation, we have

$$I_1 \leq \int_0^{(k+1)/n} |\Delta_{\pm t}^k \sum_{v=1}^{\infty} \Delta^2 a_{v-1} \overline{F}_{v-1}(x)| dx$$

$$\leq \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_{\pm \alpha t}^{(k+1)/n \pm \alpha t} \left| \sum_{v=1}^{\infty} \Delta^2 a_{v-1} \frac{\sin vx}{4 \sin^2 x/2} \right| dx$$

$$< 2 \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_0^{(2k+1)/n} \left| \sum_{v=1}^{\infty} \Delta^2 a_{v-1} \frac{\sin vx}{4 \sin^2 x/2} \right| dx$$

$$= 2^{k+1} \int_0^{(2k+1)/n} \left| \sum_{v=1}^{\infty} \Delta^2 a_{v-1} \frac{\sin vx}{4 \sin^2 x/2} \right| dx.$$

Since $\frac{1}{4 \sin^2 x/2} = 1/x^2 + O(1)$ for small x, we have

$$I_1 \leq C_k \sum_{m=n}^{\infty} \int_{(2k+1)/(m+1)}^{(2k+1)/n} \left| \sum_{v=1}^{\infty} \Delta^2 a_{v-1} \frac{\sin vx}{x^2} \right| dx + C_k A_n.$$

But in the subinterval $\left[\frac{2k+1}{m+1}, \frac{2k+1}{m}\right] (m = n, n+1, n+2 \cdots)$, we have

$$\sum_{v=1}^{\infty} \Delta^2 a_{v-1} \frac{\sin vx}{x^2}$$

$$= \sum_{v=1}^{m} \Delta^2 a_{v-1} \frac{vx}{x^2} + O\left(\sum_{v=1}^{m} \left| \Delta^2 a_{v-1} \right| \frac{v^3 x^3}{x^2} + \sum_{v=m+1}^{\infty} \left| \Delta^2 a_{v-1} \right| x^{-2} \right)$$

$$= (-a_m - m \Delta a_m)/x + O\left(\sum_{v=1}^{m} \left| \Delta^2 a_{v-1} \right| \frac{v^3}{m} + \sum_{v=m+1}^{\infty} \left| \Delta^2 a_{v-1} \right| m^2 \right)$$

$$= -(m+1) a_m + O\left(\left| a_m \right| + m^2 \left| \Delta a_m \right| + \sum_{v=1}^{m} v^2 \left| \Delta^2 a_{v-1} \right| + \sum_{v=m+1}^{\infty} m^2 \left| \Delta^2 a_{v-1} \right| \right).$$

Thus $I_1 \leq C_k \sum_{m=n}^{\infty} m^{-1}|a_m| + O\Big(\sum_{m=n}^{\infty} m^{-2}|a_m| + \sum_{m=n}^{\infty}|\Delta a_m| + \sum_{m=n}^{\infty} m^{-2}\sum_{v=1}^{m} v^2|\nabla^2 a_{v-1}|$

$$+\sum_{m=n}^{\infty}\sum_{v=m+1}^{\infty}|\Delta^2 a_{v-1}|\Big) + O(A_n).$$

Since $a_v \to 0$, we have

$$|a_m| = |\sum_{k=m}^{\infty}\sum_{i=k}^{\infty}\Delta^2 a_i|$$

$$\leq \sum_{k=m}^{\infty}\sum_{i=k}^{\infty}|\Delta^2 a_i|$$

$$\leq \sum_{i=m}^{\infty} i|\Delta^2 a_i|,$$

and therefore

$$\sum_{m=n}^{\infty} m^{-2}|a_m| = O(A_n).$$

Moreover,

$$\sum_{m=n}^{\infty}|\Delta a_m| \leq \sum_{m=n}^{\infty}\sum_{v=m+1}^{\infty}|\Delta^2 a_{v-1}| \leq A_n$$

and

$$\sum_{m=n}^{\infty} m^{-2}\sum_{v=1}^{m} v^2|\Delta^2 a_{v-1}| = \sum_{v=1}^{n} v^2|\Delta^2 a_{v-1}|\sum_{m=n}^{\infty} m^{-2} + \sum_{v=n+1}^{\infty} v^2|\Delta^2 a_{v-1}|\sum_{m=v}^{\infty} m^{-2}$$

$$= O(A_n).$$

Hence it follows that

$$(4.2) \qquad I_1 \leq C_k \sum_{v=n}^{\infty} v^{-1}|a_v| + C_k A_n.$$

To estimate I_2, we apply Abel's transformation and then using Lemma 1, we obtain

$$(4.3)\ I_2 = \int_{(k+1)/n}^{\pi} |\sum_{v=1}^{\infty}\Delta^2 a_{v-1}\Delta_{\pm t}^{k}\bar{K}_{v-1}(x)|dx$$

$$\leq \Big[\sum_{v=1}^{n} + \sum_{v=n+1}^{\infty}\Big] v|\Delta^2 a_{v-1}| \int_{(k+)/n}^{\pi} |\Delta_{\pm t}^{k}\bar{K}_{v-1}(x)|dx$$

$$\leq C_k\Big[n^{-k}\sum_{v=1}^{n} v^{k+1}|\Delta^2 a_{v-1}| + \sum_{v=n+1}^{\infty} v|\Delta^2 a_{v-1}|\Big].$$

$$= C_k B_n.$$

The result follows from (4.1), (4.2) and (4.3).

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