Kyungpook Math. J.
Volume 18, Number 2
December, 1978

ON SOME TYPES OF AFFINE MOTIONS IN AFFINELY CONNECTED GENERALISED 2-RECURRENT SPACES-III

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1. Introduction

We consider a non-flat affinely connected generalised 2-recurrent space $AG\{2K_N\}$ with symmetric connection Γ^i_{jk}, curvature tensor $B^i_{jkl}(=-B^i_{jlk})$ and with β_k and a_{mn} as its respective vector and tensor of recurrence. Then

(1) $$\nabla_n\nabla_m B^i_{jkl}=a_{mn}B^i_{jkl}+\beta_n\nabla_m B^i_{jkl}$$

where ∇ denotes covariant differentiation with respect to Γ^i_{jk}.
Further, let the space admit an infinitesmall co-ordinate transformation.
(1′) $\bar{x}^i=x^i+\xi^i(x)\delta t$ [δt being infinitesimal constant] satisfying the condition

(2) $$\pounds\Gamma^i_{jk}=\nabla_k\nabla_j\xi^i+B^i_{jkl}\xi^l=0,$$

where $\pounds$ denotes Lie-derivative with respect to the above transformation. Such transformations are called *affine motions.*

It is known [1] that the following relations hold in an $AG\{2K_N\}$ admitting affine motions:

(3) $$C\cdot\nabla_m B^i_{jkl}+(\nabla_m C-C\beta_m-\xi^t a_{tm})B^i_{jkl}=\nabla_m\xi_t\cdot\nabla_t B^i_{jkl}$$

(4) $$\xi^t\nabla_t B^i_{jkl}=CB^i_{jkl},\ \text{where } C=f^{mn}A_{mn} \text{ and } A_{mn}=a_{mn}-a_{nm}.$$

(5) $$\pounds B^i_{jkl}=0=\xi^t\nabla_t B^i_{jkl}-B^t_{jkl}\nabla_t\xi^i+B^i_{tkl}\nabla_j\xi^t+B^i_{jtl}\nabla_k\xi^t+B^i_{jkt}\nabla_l\xi^t$$

(6) $$\pounds c=\xi^m\nabla_m c=0.$$

In an earlier paper [2] affine motions generated by a bi-recurrent vector field have been studied in an $AG\{2K_N\}$. The object of the present paper is to obtain further results on such affine motions generated by a bi-recurrent vector field.

2. Affine motions corresponding to a bi-recurrent vector field

We consider an affine motion generated by a bi-recurrent vector field ξ^i given by (7) $\nabla_n\nabla_m\xi^i=S_{mn}\,\xi^i$ where $S_{mn}\neq a_{mn}$. Putting $\nabla_j\xi^i=B^i_{jkl}f^{kl}$ where f^{kl} is a skew-symmetric tensor it follows from (2) and (7) that

$$(8) \qquad \xi^j\nabla_j\xi^t=\xi^jB^t_{jmn}f^{mn}=f^{mn}(-B^t_{mnj}+B^t_{nmj})\xi^j=f^{mn}[(\nabla_n\nabla_m-\nabla_n\nabla_m)\xi^t]$$
$$=f^{mn}(S_{mn}-S_{nm})\xi^t=F\xi^t \text{ where } F=f^{mn}(S_{mn}-S_{nm}).$$

From (1′), (2) and (7) it follows that

$$(9) \qquad S_{jk}\xi^i+B^i_{jkl}\xi^l=0.$$

Multiplying (9) by ξ^k and summing over k, using $B^i_{jkl}\xi^k\xi^l=0$, we get

$$(10) \qquad S_{jk}\xi^k=0.$$

Also, putting $i=k$, and summing over k in (9) and using (10) we obtain

$$(11) \qquad B_{jt}\xi^t=0.$$

Further, we know that $B^h_{hkl}=-B_{kl}+B_{lk}$. Hence the contraction on i and j in (2) yield $\nabla_k\alpha=(B_{kl}-B_{lk})\xi^l$ where $\alpha\overset{\text{def}}{=}\nabla_t\xi^t$, whence from (11) we get

$$(12) \qquad \nabla_k\alpha=-B_{tk}\xi^t.$$

Operating $\nabla_l\nabla_j$ on (12) we get

$$\nabla_l\nabla_j\nabla_k\alpha=-\nabla_l\xi^t\cdot\nabla_jB_{tk}-\nabla_lB_{tk}\cdot\nabla_j\xi^t-\xi^t(\beta_l\nabla_jB_{tk}+a_{jl}B_{tk})-S_{jl}\cdot B_{tk}\xi^t.$$

Making a commutator on j and 1, from the last formula, we get,

$$B^t_{jkl}\cdot\nabla_t\alpha=(A_{lj}+S_{lj}-S_{jl})B_{tk}\xi^t-\xi^t(\beta_l\nabla_jB_{tk}-\beta_j\nabla_lB_{tk}).$$

Multiplying this by ξ^l and summing over 1, using (9) and $\xi^t\nabla_t\alpha=0$ we obtain

$$[(A_{lj}+S_{lj}-S_{jl})\xi^l]\; B_{tk}\xi^t=\xi^t\xi^l(\beta_l\nabla_jB_{tk}-\beta_j\nabla_lB_{tk}).$$

Using (4) it follows from the above equation that,

$$(13) \qquad [A_{lj}+S_{lj}-S_{jl})\xi^l+C\beta_j]\,B_{tk}\xi^t=d\xi^t\nabla_jB_{tk} \text{ where } d=\xi^t\beta_t.$$

Let us suppose that $d=0$, i.e. the vector of recurrence β_t of the space is pseudo-orthogonal to the vector ξ^t generating the motion. Hence, from(13) it follows that we have the following two cases:

Case I. $B_{tk}\xi^t=0$.

Case II. $(A_{lj}+S_{lj}-S_{jl})\xi^l+C\beta_j=0$.

Case I. In this case we have not only $\nabla_k\alpha=\nabla_k\nabla_t\xi^t=-B^t_{tkl}\xi^l=(B_{kl}-B_{lk})\xi^l=0$

but also,

(14) $$S_{jk}\xi^k=S_{jk}\xi^j=0.$$

Using the formula[[3], (2.6)]

(15) $$A_{mt}B^t_{jkl}=a_{lt}B^t_{mkj}-a_{kt}B^t_{mlj}-a_{lj}B_{mk}+a_{kj}B_{ml}-A_{kl}A_{mj}+\beta_k\nabla_l A_{jm}-\beta_l\nabla_k A_{jm}.$$

Multiplying(15) by ξ^m and writing $A_{mt}\xi^m=\eta_t$, we get

(16) $$\eta_t B^t_{jkl}=(S_{kj}-S_{jk})a_{lt}\xi^t+(S_{jl}-S_{lj})\ a_{kt}\xi^t-A_{kl}\eta_j+\xi^m(\beta_k\nabla_l A_{jm}-\beta_l\nabla_k A_{jm}).$$

Multiplying (16) by ξ^l and using $\eta_t B^t_{jkl}\xi^l=-\eta_t\nabla_k\nabla_j\xi^j=-\eta_t S_{ik}\xi^t=0$

($\because\ \eta_t\xi^t=0$) and (15), we get

(17) $$\eta_j\eta_k=(S_{jk}-S_{kj})a_{lt}\xi^l\xi^t-\beta_k\xi^m\xi^t\nabla_l A_{jm}.$$

Also, we know that [[3], (2.1)]

(18) $$A_{mn}B^i_{jkl}=B^i_{tmn}B^t_{jkl}-B^t_{jmn}B^i_{tkl}+B^t_{kmn}B^i_{jlt}-B^t_{lmn}B^i_{jkt}+\beta_m\nabla_n B^i_{jkl}-\beta_n\nabla_m B^i_{jkl}.$$

First operating ∇_p on (18) and then transvecting with $\xi^p\xi^n$, we get

(19) $$\xi^p\xi^n\nabla_p A_{mn}=C\eta_m+\beta_m a_{pn}\xi^p\xi^n+C\xi^p\nabla_p\beta_m-2C^2\beta_m.$$

Using (19) equation (17) reduces to

(20) $$\eta_j(\eta_k+C\beta_k)=(S_{jk}-S_{kj}-\beta_j\beta_k)a_{lt}\xi^l\xi^t+2C^2\beta_j\beta_k-C\beta_k\xi^p\nabla_p\beta_j.$$

Interchanging j and k and subtracting the resulting equation from it, we get

$$C(\eta_j\beta_k-\eta_k\beta_j)=2(S_{jk}-S_{kj})a_{lt}\xi^l\xi^t+C\xi^p(\beta_j\nabla_p\beta_k-\beta_k\nabla_p\beta_j).$$

Multiplying this equation by f^{jk} and summing over j and k, we get

$$0=2F\ a_{lt}\xi^l\xi^t.$$

Since $F\neq0$, we obtain, therefore,

(21) $$a_{lt}\xi^l\xi^t=0.$$

Transvecting equation (3) with ξ^m and using (4), (6), (8) and $d=0$, we obtain

$$[C\cdot(C-F)-\xi^t\xi^m a_{tm}]B^i_{jkl}=0$$

which reduces in view of (21) to $C\cdot(C-F)\cdot B^i_{jkl}=0$.

But since $C\neq0$, $C\neq F$ and $B^i_{jkl}\xi^l\neq0$, the last relation is an impossible one.

Hence, we deduce the following theorems:

THEOREM 1. *If in an $AG\{2K_N\}$ the vector of recurrence of the space is pseudo-orthogonal to the vector ξ^t generating an affine motion of the bi-recurrent from given by $\nabla_k\nabla_j\xi^i=S_{jk}\xi^i(S_{jk}\neq a_{jk})$ then $B_{tk}\xi^t\neq 0$.*

THEOREM 2. *If an $AG\{2K_N\}$ admit an affine motion generated by a bi-recurrent vector field ξ^t and $B_{tk}\xi^t=0$, then the vector ξ^t cannot be pseudo-orthogonal to the vector of recurrence of the space.*

Case II. In this case by virtue of (10), we have

$$\eta_j+S_{lj}\xi^l+C\beta_j=0. \tag{22}$$

Since $S_{jl}\xi^j=\nabla_l\nabla_j\xi^j=\nabla_l\alpha$, the last formula can be written as $\eta_j+\nabla_j\alpha+C\beta_j=0$ which reduces by virtue of (12) to

$$\eta_l-B_{jl}\xi^j+C\beta_l=0. \tag{23}$$

Therefore, it follows from (22) and (23) that

$$(B_{jl}+S_{jl})\xi^j=0. \tag{24}$$

Differentiating (24) covariantly twice, we have

$$\xi^m(\nabla_n\nabla_kS_{ml}+a_{kn}B_{ml}+\beta_n\nabla_kB_{ml})+\nabla_n(B_{ml}+S_{ml})\cdot\nabla_k\xi^m$$
$$+(B_{ml}+S_{ml})S_{kn}\xi^m+\nabla_n\xi^m\cdot\nabla_k\,(B_{ml}+S_{ml})=0.$$

Making a commutator on n and k and using(24) we find

$$A_{kn}B_{ml}\xi^m+\xi^m(\beta_n\nabla_kB_{ml}-\beta_k\nabla_nB_{ml})-B^t_{lkn}S_{mt}\xi^m-S_{tl}(-B^t_{knm}+B^t_{nkm})\xi^m=0.$$

Using (2) this reuces to

$$A_{kn}B_{ml}\xi^m+\xi^m(\beta_n\nabla_kB_{ml}-\beta_k\nabla_nB_{ml})-B^t_{lkn}S_{mt}\xi^m-S_{kn}S_{tl}\xi^t+S_{nk}S_{tl}\xi^t=0.$$

Using (24) in the last equaton, we get

$$(A_{kn}+S_{kn}-S_{nk})(\eta_l+C\beta_l)+B^t_{lkn}(\eta_t+C\beta_t)+\xi^m(\beta_n\nabla_kB_{ml}-\beta_k\nabla_nB_{ml})=0.$$

Multiplying this by $\xi^k$and summing over k, using $d=0$, $\eta_t\xi^t=0$ and

$B^t_{lkn}\xi^k=S_{ln}\xi^t$, we get, $[(A_{kn}+S_{kn}-S_{nk})\xi^k](\eta_l+C\beta_l)+C\beta_nB_{ml}\xi^m=0$,

which reduces by virtue of (23) to $[(A_{kn}+S_{kn}-S_{nk})\xi^k+C\beta_n](\eta_l+C\beta_l)=0$.

Since in this case $B_{tk}\xi^t\neq 0$ it follows from (23) and (12) that $\eta_l+C\beta_l\neq 0$. Contracting i and l in (5) we obtain

$$0=\pounds B_{jk}=CB_{jk}+B_{tk}\nabla_j\xi^t+B_{jt}\nabla_k\xi^t.$$

Operating ∇_l to the last formula and using $B_{jt}\nabla_l\nabla_k\xi^t=B_{jt}\cdot S_{kl}\xi^t=0$ obtained from (7) and (17) we get

$$B_{jk}\cdot\nabla_l C+C\cdot\nabla_l B_{jk}+\nabla_l B_{tk}\cdot\nabla_j\xi^t+S_{jl}\cdot B_{tk}\xi^t+\nabla_l B_{jt}\cdot\nabla_k\xi^t=0.$$

Using (12) it follows from the above equation that

$$B_{jk}\cdot\nabla_l C+C\cdot\nabla_l B_{jk}+\nabla_l B_{tk}\cdot\nabla_j\xi^t+\nabla_l B_{jt}\cdot\nabla_k\xi^t-S_{jl}\cdot\nabla_k\alpha=0.$$

Operation of ∇_m on the last formula and use of this formula and

$$a_{lm}\mathcal{L}B_{jk}=0 \quad \text{yield}$$

$$B_{jk}\cdot\nabla_m\nabla_l C+\nabla_m B_{jk}\cdot\nabla_l C+\nabla_m C\cdot\nabla_l B_{jk}+S_{jm}\xi^t\nabla_l B_{tk}+S_{km}\xi^t\nabla_l B_{jt}$$
$$-\nabla_m S_{jl}\cdot\nabla_k\alpha-S_{jl}\cdot\nabla_m\nabla_k\alpha+\beta_m S_{jl}\cdot\nabla_k\alpha-\beta_m B_{jk}\cdot\nabla_l C=0.$$

Multiplying this equation by ξ^l and using (4), (6), (10), (11), (12), we get

$$(\xi^l\nabla_m\nabla_l C+C\cdot\nabla_m C)B_{jk}=(CS_{jm}+\xi^l\nabla_m S_{jl})\cdot\nabla_k\alpha.$$

Since $\nabla_m\mathcal{L}C=\nabla_m(\xi^l\nabla_l C)=\xi^l\nabla_m\nabla_l C+\nabla_l C\cdot\nabla_m\xi^l=0$ the above equation reduces to

(25) $$(C\cdot\nabla_m C-\nabla_l C\cdot\nabla_m\xi^l)B_{jk}=(CS_{jm}+\xi^l\nabla_m S_{jl})\cdot\nabla_k\alpha.$$

If possible, let

(26) $$C\cdot\nabla_m C-\nabla_l C\cdot\nabla_m\xi^l=0.$$

Then from (25) we get $CS_{jm}+\xi^l\nabla_m S_{jl}=0$.

Making use of (10) this can be expressed as $CS_{jm}=S_{jl}\cdot\nabla_m\xi^l$ which reduces on transvection with ξ^j to

(27) $$C\cdot\nabla_m\alpha=\nabla_t\alpha\cdot\nabla_m\xi^t.$$

Multiplying (27) by a scalar function $C(\neq 0)$ and differentiating the last relation covariantly, using $C\nabla_t\alpha\cdot\nabla_p\nabla_m\xi^t=CS_{mp}\xi^t\nabla_t\alpha=0$, we get

(28) $$2C\cdot\nabla_p C\cdot\nabla_m\alpha+C^2\nabla_p\nabla_m\alpha=C\cdot\nabla_p\nabla_t\alpha\cdot\nabla_m\xi^t+C\cdot\nabla_p C\cdot\nabla_t\alpha\cdot\nabla_m\xi^t.$$

Contracting (28) with ξ^m and using (6), (8), we get

(29) $$C^2\xi^m\nabla_p\nabla_m\alpha=C\cdot F\cdot\xi^t\nabla_p\nabla_t\alpha \quad [\because\ F\cdot\nabla_p C\cdot\xi t\nabla t\alpha=0];$$

on the other hand

$$\xi^t\nabla_m\nabla_t\alpha=\nabla_m(\xi^t\nabla_t^{\alpha})-\nabla_t\alpha\cdot\nabla_m\xi^t=-\nabla_t\alpha\cdot\nabla_m\xi^t;$$

whence (29) reduces to $C\cdot(C-F)\cdot\nabla_m\alpha\cdot\nabla_p\xi^m=0$. But in general $C\neq 0$, $C\neq F$, therefore, $\nabla_m\alpha\cdot\nabla_p\xi^m=0$. Using this into (27) we have $C\cdot\nabla_p\alpha=0$. Since $C\neq 0$, we obtain, therefore, $\nabla_p\alpha=0$ or $B_{tk}\xi^t=0$ (by (12)) which contradicts our assumption that $B_{tk}\xi^t\neq 0$. So, $C\cdot\nabla_m C-\nabla_l C\cdot\nabla_m\xi^l\neq 0$;

whence, from (25) we can express the Ricei-tensor B_{jk} as

(30) $$B_{jk}=\lambda_j\cdot\nabla_k\alpha.$$

Thus, we have the following theorem:

THEOREM 3. *If in an $AG\{2K_N\}$ the vector of recurrence β_t is pseudo-orthogonal to the vector ξ^t generating an affine motion of bi-recurrent form given by $\nabla_k\nabla_j\xi^i=S_{jk}\xi^i$ $(S_{jk}\neq a_{jk})$ then the Ricci-tensor B_{jk} can be resolved as $\lambda_j\cdot\nabla_k\alpha$ where $\alpha=\nabla_t\xi^t$ with the use of a covariant vector λ_j, provided $(A_{mn}+S_{mn}-S_{nm})\xi^m+C\beta_n=0$.*

3. Some consequences

If $\beta_n=0$, the space $AG\{2K_N\}$ becomes a bi-recurrent space. In this case d is automatically zero. Hence, theorems (1) and (2) of Takano and Imai [4] become particular cases of our theorems (1) and (3).

Acknowledgement In conclusion, I acknowledge my grateful thanks to prof. M. C. Chaki for suggesting the problem and for his help and guidance in the preparation of this paper.

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