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ON A FUNDAMENTAL THEOREM FOR EXPECTATION

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It is a fundamental fact in probability theory that the abstract integral of a random variable with respect to a probability measure can be expressed as a Lebesgue Stieltjes integral on the real line [1, Proposition a, p. 166]. This result has an analogue for random vectors which assume values in the real n-dimensional Euclidean space R^n [1, p. 169]. In this note, via the notion of equidistributivity, we give a unified approach to these results, showing that these two results, one seemingly more general than the other, are in fact equivalent.

In what follows, inequalities between n-vectors in R^n are defined componentwise.

For any probability space $(\Omega, \mathscr{A}, P)$, we denote by $M(\Omega, \mathscr{A}, P)$ the collection of all real valued finite random variables on $(\Omega, \mathscr{A}, P)$. If $n \geq 1$ is an integer, let $M^n(\Omega, \mathscr{A}, P)$ denote the Cartesian product of $M(\Omega, \mathscr{A}, P)$ with itself n times. Let $(\Omega', \mathscr{A}', P')$ be another probability space. Two random vectors $\underset{\sim}{X} \in M^n(\Omega, \mathscr{A}, P)$ and $\underset{\sim}{Y} \in M^n(\Omega', \mathscr{A}', P')$ are said to be *equidistributed* (written $\underset{\sim}{X} \sim \underset{\sim}{Y}$) with respect to the probability measures P and P' whenever they have the same joint distribution functions, i.e.,

$$(1) \qquad P[\{\omega : \underset{\sim}{X}(\omega) \leq \underset{\sim}{x}\}] = P'[\{\omega : \underset{\sim}{Y}(\omega) \leq \underset{\sim}{x}\}]$$

for all n-vectors $\underset{\sim}{x} \in R^n$.

It is well known [1, p. 166 and p. 169] that any given random vector $\underset{\sim}{X} \in M^n(\Omega, \mathscr{A}, P)$ induces on its range space a corresponding probability space $(R^n, \mathscr{B}^n, P_{\underset{\sim}{X}})$, where $\mathscr{B}^n$ denotes the σ-field of all the n-dimensional Borel subsets of R^n, and $P_{\underset{\sim}{X}} : \mathscr{B}^n \to [0,1]$ (called the *probability distribution of* $\underset{\sim}{X}$ [1, p. 166]) is the probability measure defined by

$$(2) \qquad P_{\underset{\sim}{X}}[B] = P[\underset{\sim}{X} \in B]$$

for all $B \in \mathscr{B}^n$.

THEOREM 1. *Two random vectors* $\underset{\sim}{X} \in M^n(\Omega, \mathscr{A}, P)$ *and* $\underset{\sim}{Y} \in M^n(\Omega', \mathscr{A}', P')$ *are equidistributed if and only if*

$$(3) \qquad P[\underset{\sim}{X} \in B] = P'[\underset{\sim}{Y} \in B]$$

for all n-dimensional Borel subsets $B\subset R^n$, i.e., if and only if they induce the same probability distributions on R^n.

PROOF. Clearly, the condition is sufficient.

Conversely, assume that $\underset{\sim}{X}\sim\underset{\sim}{Y}$. Let $\mathscr{C}$ denote the collection of all finite disjoint unions of sets of the form $[\underset{\sim}{a}, \underset{\sim}{b})=\{\underset{\sim}{x}\in R^n : \underset{\sim}{a}\leq\underset{\sim}{x}<\underset{\sim}{b}\}$, where $\underset{\sim}{a}, \underset{\sim}{b}\in R^n$, $\underset{\sim}{a}<\underset{\sim}{b}$. Then it is easy to see that $\mathscr{C}$ is a field (that is, closed under finite unions and complementation) and that it generates the Borel σ-field $\mathscr{B}^n$ of R^n. Let $P_{\underset{\sim}{X}}$ and $P_{\underset{\sim}{Y}}$ be the probability distributions of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ respectively, which are defined as in (2). Since $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are equidistributed, it is not hard to see that $P_{\underset{\sim}{X}}$ and $P_{\underset{\sim}{Y}}$ agree on sets of the form $[\underset{\sim}{a}, \underset{\sim}{b})$, where $\underset{\sim}{a}, \underset{\sim}{b}\in R^n$, $\underset{\sim}{a}<\underset{\sim}{b}$, and hence on all the sets in $\mathscr{C}$, by the additivity property of measures. Thus, by the extension theorem for measures [1, Theorem A, p. 87], we conclude that $P_{\underset{\sim}{X}}$ and $P_{\underset{\sim}{Y}}$ agree on all the Borel sets in $\mathscr{B}^n$, i.e., condition (2) holds.

COROLLARY 2. *If $\underset{\sim}{X}\in M^n(\Omega, \mathfrak{A}, P)$ and $\underset{\sim}{Y}\in M^n(\Omega', \mathfrak{A}', P')$ are equidistributed, then*

$$f(\underset{\sim}{X})\sim f(\underset{\sim}{Y}) \tag{4}$$

for all Borel measurable functions $f: R^n\to R^m$ where m is any natural number not necessarily distinct from n.

PROOF. Let $B\subset R^m$ be an m-dimensional Borel set. Then $f^{-1}(B)\subset R^n$ is an n-dimensional Borel set, and so

$$P[\underset{\sim}{X}\in f^{-1}(B)]=P'[\underset{\sim}{Y}\in f^{-1}(B)] \text{ or } P[f(\underset{\sim}{X})\in B]=P'[f(\underset{\sim}{Y})\in B].$$

Hence $f(\underset{\sim}{X})\sim f(\underset{\sim}{Y})$, by Theorem 1.

COROLLARY 3. *Let $\underset{\sim}{X}\in M^n(\Omega, \mathfrak{A}, P)$ be any random variable and let $I: R^n\to R^n$ be the identity map of R^n. Then*

$$\underset{\sim}{X}\sim I \tag{5}$$

whenever R^n is provided with the probability (distribution) $P_{\underset{\sim}{X}}$ (of $\underset{\sim}{X}$) on its Borel σ-field $\mathscr{B}^n$.

Thus, if $f: R^n\to R^m$ is any Borel measurable function, then

$$f(\underset{\sim}{X})\sim f \tag{6}$$

with respect to the probability measures P and $P_{\underset{\sim}{X}}$.

PROOF. The assertion (5) is an immediate consequence of Theorem 1 and the definition of $P_{\underset{\sim}{X}}$, i.e., (3) and (2). With (5), the assertion (6) then follows directly from Corollary 2.

THEOREM 4. *If $X \in M(\Omega, \mathfrak{A}, P)$ and $Y \in M(\Omega', \mathfrak{A}', P')$ are random variables which are equidistributed, then $E[X] = E[Y]$ in the sense that both sides may be infinite and that if either side is finite, so is the other and they are equal.*

PROOF. Since it is clear from Corollary 2 that $X \sim Y$ implies both $X^+ \sim Y^+$ and $X^- \sim Y^-$, where $X^+ = X \vee 0$ and $X^- = (-X)^+$, we need only prove the assertion for nonnegative random variables X and Y such that $X \sim Y$.

Suppose $X \geq 0$, $Y \geq 0$ and $X \sim Y$. Then X and Y can be approximated respectively by an increasing sequence $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ of elementary functions defined by

$$X_n = \sum_{k=1}^{\infty} \frac{k}{2^n} I_{\left\{\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right\}}$$

and similarly for Y_n, $n = 1, 2, 3, \cdots$. But Theorem 1 implies that $P\left[\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right] = P'\left[\frac{k}{2^n} < Y \leq \frac{k+1}{2^n}\right]$ for all $k, n = 1, 2, 3, \cdots$, and so $E[X_n] = E[Y_n]$. By the monotone convergence theorem, we therefore infer that $E[X] = E[Y]$.

COROLLARY 5. *If $X \in M(\Omega, \mathfrak{A}, P)$ is any given random variable and if $f: R \to R$ is any Borel measurable function, then*

$$E[f(X)] = \int_R f\, dP_X = \int_{-\infty}^{\infty} f(x) dF_X(x) \tag{7}$$

where $F_X: R \to [0, 1]$ is the distribution function of X, i.e., $F_X(x) = P[X \leq x]$, $x \in R$.

PROOF. The left-hand equality of (7) follows immediately from Corollary 3 and Theorem 4.

The right-hand equality of (7) is a consequence of the fact that P_X is the Lebesgue Stieltjes measure on R generated by F_X [1, p. 167].

COROLLARY 6. *If $\underset{\sim}{X} \in M^n(\Omega, \mathfrak{A}, P)$ is a random vector and if $f: R^n \to R$ is any Borel measurable function, then*

$$E[f(\underset{\sim}{X})] = \int_{R^n} f\, dP_{\underset{\sim}{X}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n) dF_{\underset{\sim}{X}}(x_1, x_2, \cdots, x_n). \tag{8}$$

PROOF. This follows as in Corollary 5 on noting that $P_{\underset{\sim}{X}}$ is the Lebesgue Stieltjes measure on R^n generated by $F_{\underset{\sim}{X}}$ [1, p. 168].

REMARK. Theorem 4 and Corollary 5 are in fact equivalent, since Corollary 5 can be established on its own as in [1, Proposition a, p. 166] and Theorem 4 can then be derived as a particular case. In this way, Corollary 6 can be regarded as a particular case (and hence an equivalent form) of Corollary 5.

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