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INTEGRALS INVOLVING SPHEROIDAL WAVE FUNCTION AND THEIR APPLICATIONS IN HEAT CONDUCTION

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0. Abstract

This paper deals with the evaluation of two definite integrals involving spheroidal wave function, H-function of two variables, and the generalized hypergeometric function. Also, an expansion formula for the product of generalized hypergeometric function and the H-function of two variables has been obtained. Since the H-function of two variables, spheroidal wave functions, and the generalized hypergeometric function may be transformed into a number of higher transcendental functions and polynomials, the results obtained in this paper include some known results as their particular cases.

As an application of such results, a problem of heat conduction in a non-homogenous bar has been solved by using the generalized Legendre transform [9].

1. Introduction

Let us abbreviate, for convenience, the p-parameter sequence $(a_1, \alpha_1), \cdots, (a_p, \alpha_p)$, and the q-parameter sequence $(b_1, \beta_1), \cdots, (b_q, \beta_q)$, by $[(a_p, \alpha_p)]$ and $[(b_q, \beta_q)]$ respectively.

We start by recalling the familiar H-function in the form (cf. [7], p. 408):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}[(a_p,\alpha_p)]\\ [(b_q,\beta_q)]\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=1+m}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=1+n}^{p}\Gamma(a_j-\alpha_j s)}\, z^s\, ds, \tag{1.1}$$

where L is a suitable contour.

The H-function of two complex variables $H(x, y)$ has been analogously defined and represented by Mittal and Gupta [11, p. 117]. For convenience, and brevity, however, we shall use the contracted notation introduced by Srivastava and Panda ([13], p. 266, Eq. (1.5) et seq.) throughout the present paper.

The known results ([8], p. 16; [5], p. 316; [10], p. 33) required in the sequel may be recalled as follows:

(i) The spheroidal wave function can be expressed as:

$$S_{mn}(c,x)=\sum_{r=0,1}^{\infty}{}^{*}\, d_r^{mn}(c)P_{m+r}^{m}(x) \tag{1.2}$$

where the coefficients $d_r^{mn}(c)$ satisfy the recursion formula [8, eq. 3.1.4] and the asterisk* over the summation sign indicates that the sum is taken over only even or odd values of r according as $(n-m)$ is even or odd.

(ii) $\int_{-1}^{1}(1-x^2)^{\rho-1}\, P_v^m(x)\, dx$

$$=\frac{\pi 2^m\, \Gamma\left(\rho+\frac{1}{2}m\right)\, \Gamma\left(\rho-\frac{1}{2}m\right)}{\Gamma\left(1+\frac{1}{2}(v-m)\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}(v+m)\right)\Gamma\left(\rho-\frac{1}{2}v\right)\Gamma\left(1+\rho+\frac{1}{2}v\right)} \tag{1.3}$$

provided $2\,\mathrm{Re}(\rho)>|\mathrm{Re}(m)|$.

(iii) $E_a f(a)=f(a+1),\ E_a^n f(a)=E_a\,[E_a^{n-1} f(a)]$ (1.4)

where E denotes the finite difference operator. Also, we shall use the following notation throughout the paper:

$$(\alpha)_r=\frac{\Gamma(\alpha+r)}{\Gamma\alpha}=\alpha(\alpha+1)\cdots(\alpha+r-1). \tag{1.5}$$

2. Finite integrals

The main integrals to be proved here are the following:

$$I_1(\rho)=\int_{-1}^{1}(1-x^2)^{\rho-1}\, S_{mn}(c,x)\ H[y(1-x^2)^{\lambda}, z(1-x^2)^{\mu}]\,dx$$

$$=2^m\pi\sum_{r=0,1}^{\infty}{}^{*}\, d_r^{mn}(c)\left[\Gamma\left(1+\frac{1}{2}r\right)\Gamma\left(\frac{1}{2}-m-\frac{1}{2}r\right)\right]^{-1}$$

$$H_{p_1+2,q_1+2\,:\,(p_2,q_2)\,;\,(p_3,q_3)}^{0,n_1+2\,:\,(m_2,n_2)\,;\,(m_3,n_3)}\left(\begin{matrix}[(a_{p_1}:\alpha_{p_1},A_{p_1})],\\ [(b_{q_1}:\beta_{q_1},B_{q_1})],\end{matrix}\right.$$

$$\left(1-\rho+\frac{1}{2}m:\lambda,\mu\right),\ \left(1-\rho-\frac{1}{2}m:\lambda,\mu\right):[(c_{p_2},\nu_{p_2})]\,;\,[(e_{p_3},E_{p_3})]:$$

$$\left(1-\rho+\frac{1}{2}(m+r):\lambda,\mu\right),\ \left(-\rho-\frac{1}{2}(m+r):\lambda,\mu\right):[(d_{q_2},\delta_{q_2})]\,;\,[(f_{q_3},F_{q_3})]\,;$$

$$\left.y,\ z\right), \tag{2.1}$$

where λ and μ are positive integers, and

$\mathrm{Re}(\rho)>0,\ \mathrm{Re}(\rho+\lambda d_j/\delta_j+\mu f_k/F_k)>0: j=1,\cdots,n_2: k=1,\cdots,n_3:$

$|\arg y|<\frac{\pi}{2}\left[\sum_{1}^{n_1}\alpha_j-\sum_{1+n_1}^{p_1}\alpha_j+\sum_{1}^{m_2}\delta_j-\sum_{1+m_2}^{q_2}\delta_j+\sum_{1}^{n_2}\nu_j-\sum_{1+n_2}^{p_2}\nu_j-\sum_{1}^{q_1}\beta_j\right]$, *and*

$$|\arg z| < \frac{\pi}{2}\left[\sum_{1}^{n_1} A_j - \sum_{1+n_1}^{p_1} A_j + \sum_{1}^{m_3} F_j - \sum_{1+m_3}^{q_3} F_j + \sum_{1}^{n_3} E_j - \sum_{1+n_3}^{p_3} E_j - \sum_{1}^{q_1} B_j\right].$$

The series on the right hand side of (2.1) *is convergent.*

$$I_2(\rho) = \int_{-1}^{1} (1-x^2)^{\rho-1} S_{mn}(c,x)\, {}_uF_v\left[\begin{matrix}\xi_u \\ \eta_v\end{matrix} ; h(1-x^2)^k\right] H[y(1-x^2)^{\lambda}, z(1-x^2)^{\mu}]\,dx$$

$$= 2^m \pi \sum_{r=0,1}^{\infty}{}^{*} \sum_{\sigma=0}^{\infty} d_r^{mn}(c) \left[\Gamma\left(1+\frac{1}{2}r\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}r-m\right)\right]^{-1}$$

$$\frac{\prod_{j=1}^{u}(\xi_j)\,(h)^{\sigma}}{\prod_{j=1}^{v}(\eta_j)\,\sigma!} H^{0,n_1+2}_{p_1+2,q_1+2} \begin{matrix} : (m_2,n_2) : (m_3,n_3) \\ : (p_2,q_2) : (p_3,q_3) \end{matrix}$$

$$\left(\begin{matrix} \left[(a_{p_1} : \alpha_{p_1}, A_{p_1})\right] : \left(1-\rho-\sigma k+\frac{1}{2}m : \lambda, \mu\right), \\ \left[(b_{q_1} : \beta_{q_1}, B_{q_1})\right] : \left(1-\rho-\sigma k+\frac{1}{2}(m+r) : \lambda, \mu\right), \end{matrix}\right.$$

$$\left.\begin{matrix} : \left(1-\rho-\sigma k-\frac{1}{2}m : \lambda, \mu\right) : [(c_{p_2}, \nu_{p_2})] : [(e_{p_3}, E_{p_3})] : \\ \left(-\rho-\frac{1}{2}(m+r)-\sigma k : \lambda, \mu\right) : [(d_{q_2}, \delta_{q_2})] : [(f_{q_3}, F_{q_3})] \end{matrix} \; y, z\right) \tag{2.2}$$

where h, k are positive integers (either h or k may be zero). The result (2.2) *holds if $u \le v$ ($u=v+1$ and $|h|<1$), none of $\eta_1, \eta_2, \cdots, \eta_v$ is zero or a negative integer with the remaining conditions as stated in* (2.1). *The series on the right hand side of* (2.2) *is convergent.*

PROOF OF (2.1): To prove (2.1), we first express the spheroidal wave function $S_{mn}(c,x)$ in the series form (1.2), and the H-function of two variables $H[y(1-x^2)^{\lambda}, z(1-x^2)^{\mu}]$ in terms of double contour integrals form ([11], p. 117). Now, changing the order of integration and summation, evaluating the inner integral with the help of (1.3), and finally reinterpreting the double contour integrals thus involved by definition of H-function of two variables, we get the desired result.

Regarding the interchange of the order of integration and summation it is observed that x-integral is convergent if Re $(\rho)>0$, Re $(\rho+\lambda d_j/\delta_j+\mu f_k/F_k)>0$, $(j=1,\cdots,m_2 ; k=1,\cdots,m_3)$. The double contour integral converges under the conditions stated in (2.1). The series $\sum_{r=0,1}^{\infty *} d_r^{mn}(c)P_{m+r}^m(x)$ converges absolutely and uniformly for all finite x ([8], 16—17). Hence the interchange of order of integration and summation is justified [2, p. 504].

PROOF of 2.2 : On multiplying both sides of (2.1) by $\prod_{j=1}^{u}\Gamma(\xi_j+\delta)(h)^{\delta}\Big/\prod_{j=1}^{v}\Gamma(\eta_j+\delta)$ and applying the operator $\exp(E^k E_\delta)$ yields.

$$\exp(E^k\ E_\delta)\left[I_1(\rho)\prod_{j=1}^{u}\Gamma(\xi_j+\delta)\ h^{\delta}\Big/\prod_{j=1}^{v}\Gamma(\eta_j+\delta)\right.$$

$$=2^m\pi\ \exp(E^k E_\delta)\sum_{r=0,1}^{\infty *} d_r^{mn}(c)\left[\Gamma\left(1+\frac{1}{2}r\right)\ \Gamma\left(\frac{1}{2}-\frac{1}{2}r-m\right)\right]^{-1}$$

$$\frac{\prod_{j=1}^{u}\Gamma(\xi_j+\delta)h^{\delta}}{\prod_{j=1}^{v}\Gamma(\eta_j+\delta)}H^{0,n_1+2}_{p_1+2,q_1+2}\begin{matrix}:(m_2,n_2);(m_3,n_3)\\:(p_2,q_2);(p_3,q_3)\end{matrix}$$

$$\left(\begin{matrix}[(a_{p_1}:\alpha_{p_1},A_{p_1})],\ \left(1-\rho+\frac{1}{2}m:\lambda,\mu\right),\ \left(1-\rho-\frac{1}{2}m:\lambda,\mu\right):\\ [(b_{q_1};\beta_{q_1},B_{q_1})],\ \left(1-\rho+\frac{1}{2}(m+r):\lambda,\mu\right),\ \left(-\rho-\frac{1}{2}(m+r):\lambda,\mu\right):\end{matrix}\right.$$

$$\left.\begin{matrix}[(c_{p_2},\nu_{p_2})];[(e_{p_3},E_{p_3})];\\ [(d_{q_2},\delta_{q_2})];[(f_{q_3},F_{q_3})];\end{matrix}\ y,z\right).\tag{2.3}$$

Taking summation on both sides of (2.3) and using the definition of finite difference operator (1.4), we get

$$\sum_{\sigma=0}^{\infty}\left[\frac{\prod_{j=1}^{u}\Gamma(\xi_j+\delta+\sigma)(h)^{\delta+\sigma}}{\prod_{j=1}^{v}\Gamma(\eta_j+\delta+\sigma)\quad\sigma!}\int_{-1}^{1}(1-x^2)^{\rho+\sigma k-1}\right.$$

$$\left.S_{mn}(c,x)H[y(1-x^2)^{\lambda},\ z(1-x^2)^{\mu}]dx\right]$$

$$=2^m\pi\sum_{\sigma=0}^{\infty}\sum_{r=0,1}^{\infty *} d_r^{mn}(c)\left[\Gamma\left(1+\frac{1}{2}r\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}r-m\right)\right]^{-1}$$

$$\frac{\prod_{j=1}^{u}\Gamma(\xi_j+\delta+\sigma)}{\prod_{j=1}^{v}\Gamma(\xi_j+\delta+\sigma)\sigma!}H^{0,n_1+2}_{p_1+2,q_1+2}\begin{matrix}:(m_2,n_2);(m_3,n_3)\\:(p_2,q_2);(p_3,q_3)\end{matrix}$$

$$\left(\begin{matrix}[(a_{p_1}:\alpha_{p_1},A_{p_1})],\ \left(1-\rho-\sigma k+\frac{1}{2}m:\lambda,\mu\right),\ \left(1-\rho-\sigma k-\frac{1}{2}m:\lambda,\mu\right):\\ [(b_{q_1}:\beta_{q_1},B_{q_1})],\ \left(1-\rho-\sigma k+\frac{1}{2}(m+r):\lambda,\mu\right),\ \left(-\rho-\frac{1}{2}(m+r)-\sigma k:\lambda,\mu\right):\end{matrix}\right.$$

$$\begin{matrix}[(c_{p_2},\nu_{p_2})] & ; [(e_{p_3},E_{p_3})] ; & \\ [(d_{q_2},\delta_{q_2})] & ; [(f_{q_3},F_{q_3})] ; & \end{matrix} \quad y, z\Bigg). \tag{2.4}$$

Now, changing the order of integration and summation on the left hand side of (2.4) which is justified [3, p.173], using the result (1.5) and, finally, replacing $\xi_j+\delta$ by ξ_j and $\eta_j+\delta$ by η_j enable us to obtain the value of the integral (2.2).

3. An expansion formula

In this section we derive the following expansion formula:

$$(1-x^2)^{\rho-1}\ {}_uF_v\left[\begin{matrix}\xi_u\\ \eta_v\end{matrix} ;\ h(1-x^2)^k\right] H\left[y(1-x^2)^\lambda, z(1-x^2)^\mu\right]$$

$$= \sum_{n=0}^{\infty} I_2(\rho) S_{mn}(c,x), \tag{3.1}$$

which is valid under the same conditions as given in (2.2) ***with*** $\rho\geq 1$. $I_2(\rho)$ ***is the value of the integral defined by*** (2.2). ***The series on the right hand side of*** (3.1) ***is convergent.***

PROOF. From the general theory of Sturm-Liouville differential equations, it follows that the functions $S_{mn}(c,x)$ form the countably infinite orthonormal set complete in $(-1,1)$. Hence any arbitrary function $f(x)\in(-1,1)$ can be represented as a linear combination of these functions, i.e.

$$f(x)=(1-x^2)^{\rho-1}\ {}_uF_v\left[\begin{matrix}\xi_u\\ \eta_v\end{matrix},\ h(1-x^2)^k\right]H\left[y(1-x^2)^\lambda, z(1-x^2)^\mu\right]$$

$$=\sum_{n=0}^{\infty} A_n\ S_{mn}(c,x), \qquad -1<x<1. \tag{3.2}$$

[Following Churchill (1963) p.57, Taylor (1963) p.111].

On multiplying both sides of (3.2) by $S_{mn'}(c,x)$, integrating with respect to x over the interval $(-1,1)$, and making use of the orthogonality property of spheroidal wave functions [8, p.22 eqs. (3.1.32), (3.1.33)]

$$I_2(\rho) = A_n \int_{-1}^{1} [S_{mn}(c,x)]^2\, dx, \text{ for } n'=n \tag{3.3}$$

because all other terms on the right hand side of (3.3) vanish except for $n'=n$. Now, in order to avoid undesirable consequences in applications, we shall normalize the functions $S_{mn}(c,x)$ by the stipulation that

$$\int_{-1}^{1} [S_{mn}(c,x)]^2\, dx=1, \quad [(n-m) \text{ is even or odd}]$$

for all real values of c.
Hence,

$$A_n = I_2(\rho). \tag{3.4}$$

Thus, by virtue of (3.2) and (3.4), the desired expansion formula (3.1) follows.

REMARK. Regarding the convergence of the series on the right hand sides of the results (2.1), (2.2) and (3.1), it would be worth-mentioning that the ratio d_{r+2}^{mn}/d_r^{mn} is $-c^2/4r^2$ [7, p. 17] and the ratio of the gammas involving r, σ is bounded for large values of r (even or odd) and σ by virtue of the fairly well known result (cf., 6 ; p. 47) :

$$\frac{\Gamma(r+\alpha)}{\Gamma(r+\beta)} = r^{\alpha-\beta}[1+0(r^{-1})], \quad r\to\infty.$$

Hence the series on the right hand side of (2.1), (2.2) and (3.1) are uniformly and absolutely convergent by M-test.

4. Special cases

Although a lage number of new and known results can be deduced from our results by suitably specializing the parameters, but we mention here only two of them for lack of space.

(i) On setting $p_2=p_3=0$, $q_2=q_3=0$, $n_2=n_3=0$, $c=0$, $\mu=m_3=1$, $r=n$, all A_{p_1}, $B_{q_1}, E_{p_3}, F_{q_3}$ equal to unity, $f_1=0$, $z\to 0$, (2.2) reduces to the result obtained by Singh and Verma ([12], p. 325—32).

(ii) If we put $h=0$, $c=0$, $z\to 0$ in (3. 1), it reduces to the result obtained by Singh and Verma [12] with proper choice of parameters.

5. Problem of heat absorption inside the sphere

In this section, the problem of determining a function $\phi(r, x)$ which represents the temperature inside the non-homogeneous sphere $r \leq a$ is considered. The temperature on the surface $r=a$ is a prescribed function, say $f(x)$, of spherical coordinate 'x' only $(-1 \leq x \leq 1)$. Therefore, the fundamental equation of heat conduction is

$$k\left[\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial \phi}{\partial x}\right\} + \rho c_1 Q(r, x) = 0 \tag{5.1}$$

where $x=\cos\theta$ and $Q(r, x)$ is the sink of heat absorption, and k, ρ and c_1 are respectively, the conductivity, density, and specific heat of the material of the sphere.

Let

$$Q(r,x)=-\frac{k}{c_1 r^2}\left[c^2 x^2+\frac{m^2}{1-x^2}\right]\phi$$

which is linearly dependent on the temperature function $\phi(r,x)$. ***Thus,*** *the equation* (5.1) *becomes*

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\left[(1-x^2)\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}-c^2x^2-\frac{m^2}{1-x^2}\right]\phi=0. \tag{5.2}$$

Under the boundary condition:

$$\phi(a,\ x)=(1-x^2)^{\rho-1}\ H[y(1-x^2)^{\lambda},\ \ z(1-x^2)^{\mu}],\ \ -1\leq x\leq 1,$$

the solution of (5.2) *is given by*

$$\phi(r,x)=\sum_{n=0}^{\infty}I_1(\rho)S_{mn}(c,x)\left(\frac{r}{a}\right)^{\alpha}, \tag{5.3}$$

where $\alpha=-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_{mn}}$, $I_1(\rho)$ *is the value of the integral defined by* (2.1), *and the conditions of validity are the same as given in* (2.1).

PROOF. To solve the partial differential equation (5.2), we use the 'generalized Legendre transform' recently developed and defined by Gupta [9] as

$$\bar{f}_{mn}(c)=\int_{-1}^{1}F(x)S_{mn}(c,x)\ dx, \tag{5.4}$$

with the inversion formula

$$F(x)=\sum_{n=0}^{\infty}\frac{\bar{f}_{mn}(c)\ S_{mn}(c,x)}{N_{mn}}, \tag{5.5}$$

where N_{mn} is the normalization factor of $S_{mn}(c,x)$ given by Flammer [8, p. 22, eq. (3.1.33)]. It is convenient in applications to normalize the functions $S_{mn}(c,x)$ such that $N_{mn}=1$.

Now, applying the transform (5.4) to equation (5.2), we obtain

$$r_2\frac{\partial^2\bar{\phi}}{\partial r^2}+2r\frac{\partial\bar{\phi}}{\partial r}-\lambda_{mn}(c)\bar{\phi}=0 \tag{5.6}$$

where

$$\bar{\phi}=\begin{cases}\int_{-1}^{1}(r,x)S_{mn}(c,x)dx & (5.7)\\ I_1(\ ),\ \text{when}\ r=a & (5.8)\end{cases}$$

which is bounded in the region $0\leq r\leq a$.

If α,β are the roots of the indicial equation obtained after substitution $r=e^z$

in (5.6), then

$$\alpha=-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda_{mn}(c)}],$$

$$\beta=-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda_{mn}(c)}]. \tag{5.9}$$

Since the solution corresponding to the root β is inadmissible the solution of equation (5.6) is given by

$$\bar{\phi}=A_1(c)r^{\alpha}. \tag{5.10}$$

In order to determine the coefficient $A_1(c)$, we use the equation (5.8)and get

$$A_1(c)=I_1(\rho)/a^{\alpha}.$$

Hence substituting the value of $A_1(c)$ in (5.10), we get

$$\bar{\phi}=I_1(\rho)\left(\frac{r}{a}\right)^{\alpha}.$$

Finally, using the inversion formula (5.5) with the improved convension $N_{mn}=1$, we get the desired solution (5.3).

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