Linear Quadratic Regulators with Two-point Boundary Riccati Equations
(兩斷 境界 조건이 있는 리카티 식을 가진 선형 레규레이터)

權 旭 鉉*
(Kwon, Wook-Hyun)

要 約

본 논문에서는 algebraic matrix Lyapunov equations과 algebraic matrix Riccati equations에 관하여 잘 알려져 있는 중요한 결과를 확장한다. 본 연구는 matrix 미분 방정식에서 양단 경계조건이 존재하는 문제를 다루며 여기에서 얻어지는 결과는 기존하고 있는 결과를 포함하게 된다. 특히 선형 시스템이 periodic feedback gain control로 안정화 되는 필요충분조건을 구하며, two-point boundary Riccati equations의 해를 쉽게 구하는 반복 계산방법을 제시한다. 또한 interalwise receding horizon을 이용한 새로운 periodic feedback gain control이 시스템을 안정화 시켜줌을 보여준다.

Abstract

This paper extends some well-known system theories on algebraic matrix Lyapunov and Riccati equations. These extended results contain two point boundary conditions in matrix differential equations and include conventional results as special cases. Necessary and sufficient conditions are derived under which linear systems are stabilizable with feedback gains derived from periodic two-point boundary matrix differential equations. An iterative computation method for two-point boundary differential Riccati equations is given with an initial guess method. The results in this paper are related to periodic feedback controls and also to the quadratic cost problem with a discrete state penalty.

I. Introduction.

For a linear time invariant homogeneous system

$$\dot{x}(t) = Ax(t), \qquad (1.1)$$

where $x(t) \in R^n$ and A is a n x n matrix, the following well-known theory called a Lyapunov theory exists in the area of Lyapunov stability. The system (1.1) is asymptotically stable if and only if for any C such that [A, C] is observable there exists a positive difinite matrix K to the Lyapunov matrix equation

$$A'K + KA + C'C = 0. \qquad (1.2)$$

For a linear time invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad (1.3)$$

where $x(t) \in R^n$ and A and B are n x n and n x m matrices respectively, the following theory had been developed along with a linear

* 正會員, 서울大學校 計測制御工學科
(Dept. of Instrumentation and Control Engr. Seoul National Univ.)
接受日字 : 1979年 9月 15日

quadratic cost problem[2]: The system (1.3) is stabilizable if and only if for any matrix C such that [A, C] is observable there exists a positive definite matrix K to the algebraic matrix Riccati equation

$$A'K + KA - KBB'K + C'C = 0. \qquad (1.4)$$

While the computation of the solution of the Lyapunov equation (1.2) is easy since it can be transformed to a linear equation, that of the Riccati equation (1.4) is difficult since it is a nonlinear equation. Kleinman[3] suggested an iterative method from which the solution of (1.4) can be obtained by the succesive solution of the type (1.2). It is these basic well-known results that are extended in this paper. The algebraic Riccati equation (1.4) is derived from an infinite time quadratic cost problem whose optimal control is given by

$$u(t) = -B'Kx(t) \qquad (1.5)$$

where the feedback gain is constant. For feedback controls of time invariant systems it has been a usual practice to seek controls with time invariant feedback gains rather than time-varying feedback gains since time-varying gains are difficult to implement. Controls with periodic feedback gains are easily implemented in controllers with limited memory requirement. Linear homogenious systems with periodic coefficients have been demonstrated to have several similar properties to time invariant systems[11, 12]. However there are little research results on feedback control with periodic feedback gains. Recently Geering[1] considered an infinite time quadratic cost problem with a discrete state penalty

$$J(u) = \sum_{k=1}^{\infty} x'(kT)Fx(kT) + \int_0^{\infty} [x'(t)C'Cx(t) + u'(t)u(t)]\,dt \qquad (1.6)$$

for the linear system (1.3). The optimal control to the problem (1.3) and (1.6) is given by

$$u(t) = -B'K(t)x(t) \qquad (1.7)$$

where

$$-\dot{K} = A'K + KA - KBB'K + C'C \qquad (1.8)$$

$$K(kT^-) = K(kT^+) + F\ ,\ k = 0,1,2,3,4,\ldots \qquad (1.9)$$

The control law (1.7)-(1.9) has a piecewise continuous periodic feedback gain and is one of little-known periodic feedback gain controls. Implications of discrete state penalty of (1.6) in economic, political, and some engineering processes are mentioned in [1]. The results in this paper deal with differential matrix Lyapunov and Riccati equaions with two-point boundary conditions like (1.9) and include well-known results as special cases. Most results in this work will be obtained mainly from the properties on matrix equations, independently from the above optimal problem. It is noted that there are some research results on periodic open-loop control with periodic state trajectories for periodic or nonperiodic processes, for example see [13], which are different from feedback controls with periodic feedback gains.

II. Homogeneous Systems

In this section a well known Lyapunov theory on the algebraic Lyapunov matrix equation is extended to the differential Lyapunov equation with a two point boundary condition. A matrix A is said to be stable if the system (1.1) is asymptotically stable in Lyapunov sense. A stability result on linear systems is given in Lemma 2.1 as a simple form which is necessary for this paper.

Lemma 2.1. Consider a linear time-varying homogeneous systems,

$$\dot{x}(t) = A(t)x(t), \qquad (2.1)$$

where A(t) is a piecewise continuous and bounded n x n matrix. If there exists a piecewise continuous matrix W(t) such that there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ which satisfy

$$\alpha_1 |x|^2 \leq V(t,x) \triangleq x'W(t)x \leq \alpha_2 |x|^2 \qquad (2.2)$$

and there exist $\alpha_3 > 0$ and $d > 0$ which satisfy

$$V(t, x(t;x_s,t_s)) - V(t_s,x_s) \leq -\alpha_3 |x_s|^2 \qquad (2.3)$$

for $t \geq t_s + d$ and for all $t_s \in [0, \infty]$, then the system (2.1) is uniformly asymptotically stable.

Proof: Since A(t) is bounded, it follows that there exists a function $\alpha_4(\cdot)$ such that $\|\Phi(t, \tau)\| \leqslant \alpha_4(t-\tau)$ for all t, τ where $\alpha_4(\cdot)$ maps R into R and is bounded on bounded intervals. Thus $|x(t)| \leqslant \|\Phi(t, t_0)\| \cdot |x(t_0)| \leqslant \max_{t_0 \leqslant s \leqslant t_0+d} \alpha_4(s) |x(t_0)|$ for $t_0+d \geqslant t \geqslant t_0$. Also from (2.2) and (2.3) we obtain $\alpha_1 |x(t)|^2 \leqslant V(t, x(t, x(t; x_0, t_0)) \leqslant V(t_0, x_0) - \alpha_3 |x_0|^2 \leqslant V(t_0, x_0) \leqslant \alpha_2 |x_0|^2$, which impies $|x(t)|^2 \leqslant \frac{\alpha_2}{\alpha_1} |x_0|^2$ for $t \geqslant t_0 + d$. For a given ϵ, take $\delta = \min [\epsilon/\beta, \sqrt{\alpha_1}\epsilon/\sqrt{\alpha_2}]$, where $\beta \triangleq \max_{t_0 \leqslant s \leqslant t_0+d} \alpha_4(s)$. Then $|x_0| \leqslant \delta$ implies $|x(t)| \leqslant \epsilon$ for $t \geqslant t_0$. Since δ is independent of t_0, the system (2.1) is uniformly stable. Since the system (2.1) is uniformly stable, for a given η there exists a $\gamma = \gamma(\eta)$ such that $|x(t)| \leqslant \eta$ for all $t \geqslant t_0$ and $|x_0| \leqslant \gamma(\eta)$. Let a $b_0 \geqslant 0$ is given. To prove uniformly asymptotic stability it is sufficient to show that there exists $T(\eta)$ such that $|x(t)| \leqslant \gamma$ for some $t \leqslant t_0 + T$ and $|x_0| \leqslant b_0$. Suppose there is a x_0 with $|x_0| \leqslant b_0$ and $|x(t)| > \gamma$ for $t \geqslant t_0$. Then from (2.2) and (2.3) follows that $\alpha_1 |x(t_0+nd)|^2 \leqslant V(t_0+nd, x(t_0+nd; x_0, t_0)) \leqslant V(t_0, x_0) - \alpha_3 |x(t_0 + (n-1)d)|^2 - \cdots - \alpha_3 |x_0|^2 \leqslant \alpha_2 |x_0|^2 - \gamma^2 n\alpha_3$. Thus $|x(t_0+nd)|^2 \leqslant \frac{\alpha_2 b_0^2}{\alpha_1} - \frac{n\alpha_3\gamma^2}{\alpha_1}$ Let $N = \min [n \geqslant 0: \alpha_2 b_0^2/\alpha_1 - n\alpha_3\gamma^2/\alpha_1 \leqslant \gamma^2]$ and let T = Nd. Then $|x(t)| \leqslant \gamma$ for some $t \leqslant t_0 + T$, which implies that $|x(t)| \leqslant \eta$ for $t \geqslant t_0 + T$ since the system (2.1) is uniformly stable. Since T is independent of t_0, the system (2.1) is uniformly asymptotically stable. This completes the proof.

Throughout this paper $\lambda_i(A)$ denotes the eigenvalues of the matrix A and a periodic matrix $\hat{P}(t)$, $0 \leqslant t < \infty$, is defined from a given P(t) defined over $0 \leqslant t \leqslant T$ such that

$$\hat{P}(t) \triangleq \begin{cases} P(t) & 0 \leqslant t < T \\ \hat{P}(t-T). & t \geqslant T \end{cases} \tag{2.4}$$

An extended Lyapunov theory is stated in the following theorem.

Theorem 2.1. A matrix A is a stable matrix if and only if for each matrix C such that [A, C] is observable, $F \geqslant 0$, and a scalar $T > 0$, there exists a positive definite matrix solution $K(t) > 0$, $0 \leqslant t \leqslant T$, such that

$$-\dot{K}(t) = A'K(t) + K(t)A + C'C \tag{2.5}$$

$$K(T) - K(0) = F. \tag{2.6}$$

Moreover, for a stable matrix A, $K(t) \geqslant K_s$ for a $F \geqslant 0$, and $K(t) = K_s$, $0 \leqslant t \leqslant T$, if and only if $F = o$, where K_s is the solution of the algebraic Lyapunov equation (1.2).

Proof: (Necessity) The equation (2.5) can be expressed as

$$K(t) = e^{-A't}K(0)e^{-At} - \int_0^t e^{-A'(t-s)}C'Ce^{-A(t-s)}ds \tag{2.7}$$

$$= e^{A'(T-t)}K(T)e^{A(T-t)} + \int_t^T e^{A'(s-t)}C'Ce^{A(s-t)}ds. \tag{2.8}$$

From (2.6) and (2.7) we have

$$F = K(T) - K(0) = e^{-A'T}K(0)e^{-AT} - \int_0^T e^{-A'(T-s)} C'Ce^{-A(T-s)}ds - K(0) \tag{2.9}$$

which gives

$$e^{A'T}K(0)e^{AT} - K(0) = -e^{A'T}(F + \int_0^T e^{-A(T-s)} C'Ce^{-A(T-s)}ds)e^{AT} \tag{2.10}$$

The equation (2.10) is a discrete Lyapunov matrix equation and the right side of (2.10) is a positive definite matrix. Since $|\lambda_i(e^{AT})| < 1$ for a stable matrix A there exists a unique $K(0) > 0$ for the solution of (2.10) from well known results on discrete Lyapunov matrix equations. From (2.6) follows $K(T) > 0$ and from (2.8) $K(t) > 0$, $0 \leqslant t \leqslant T$. (Sufficienty) Let's take a Lyapunov function for the system (1.1) as

$$V(t,x(t)) = x'(t)\hat{K}(t)x(t). \tag{2.11}$$

Since in (2.8) $K(T) > 0$ and T is a finite value there exists positive constants α_1 and α_2 such that $\alpha_1 I \leqslant K(T) \leqslant \alpha_2 I$, $0 \leqslant tT$. This in return implies

$$\alpha_1|x(t)|^2 \leqslant V(t,x(t)) \leqslant \alpha_2|x(t)|^2. \qquad (2.12)$$

Let $\Omega = [j : jT\epsilon[t_s, t], t > t_s]$. Since $\hat{K}(t)$ is defined as in (2.4), it is piecewise absolutely continuous on the interval $[t_s, t]$ and jumps at jT, $j\epsilon\Omega$, that is, $K(jT^+) - K(jT^-) = -F$, $j\epsilon\Omega$. Thus we have

$$V(t,x(t;x_s,t_s)) - V(t_s,x_s)$$

$$= \int_{t_s}^{t} \dot{V}(\tau,x(\tau))d\tau + \sum_{j\epsilon\Omega} [V(jT^+,x(jT^+)) - V(jT^-,x(jT^-))]$$

$$= \int_{t_s}^{t} \dot{V}(\tau,x(\tau))d\tau - \sum_{j\epsilon\Omega} x'(jT)Fx(jT) \leqslant \int_{t_s}^{t} \dot{V}(\tau,x(\tau))d\tau$$

$$= \int_{t_s}^{t} \dot{x}'(\tau)\hat{K}(\tau)x(\tau) + x'(\tau)\hat{K}(\tau)\dot{x}(\tau) + x'(\tau)\dot{\hat{K}}(\tau)x(\tau)d\tau$$

$$= \int_{t_s}^{t} x'(\tau)[A'\hat{K}(\tau) + \hat{K}(\tau)A + \dot{\hat{K}}(\tau)]x(\tau)d\tau$$

$$= -\int_{t_s}^{t} x'(\tau)C'Cx(\tau)d\tau$$

$$= -x_s'\int_{t_s}^{t} e^{A'(\tau-t_s)}C'Ce^{A(\tau-t_s)}d\tau x_s$$

$$\leqslant -\alpha_3|x_s|^2, \qquad t \geqslant t_s + d, \qquad (2.13)$$

for some positive constant α_3 and an arbitrary small $d > 0$ since $[A, C]$ is observable. Thus the Lyapunov function (2.11) satisfies all conditions of Lemma 2.1. Therefore the system (1.1) is asymptotically stable. This completes the first part of the theorem. Let $\widetilde{K}(t) \doteqdot K(t) - K_s$. Then we have

$$\dot{\widetilde{K}} = \dot{K} = -A'K - KA - C'C = -A'\widetilde{K} - \widetilde{K}A \qquad (2.14)$$

and

$$\widetilde{K}(T) - \widetilde{K}(0) = F. \qquad (2.15)$$

From (2.14) and (2.15) follows that

$$\widetilde{K}(t) = e^{-A't}\widetilde{K}(0)e^{-At} \qquad (2.16)$$

$$e^{A'T}\widetilde{K}(0)e^{AT} - \widetilde{K}(0) = -F, \qquad (2.17)$$

from which follows that $K(0) \geqslant 0$ and thus $\widetilde{K}(t) \geqslant 0$, $0 \leqslant t \leqslant T$. For the case of $F = 0$ the equation (2.17) can be expressed as

$$[e^{A'T}]\widetilde{K}(0) + \widetilde{K}(0)[-e^{-AT}] = 0. \qquad (2.18)$$

Since $\lambda_i(e^{A'T}) + \lambda_j(-e^{-AT}) \neq 0$, $\widetilde{K}(0)$ is unique and thus $\widetilde{K}(0) = 0$. This implies $\widetilde{K}(t) = 0$, $0 \leqslant t \leqslant T$. This completes the proof.

Theorem 2.1 is an extension of a Lyapunov theory on (1.2) in the sense that the latter can be obtained from the former with $F = 0$. In the next section a well known theory on the algebraic Riccati matrix equation (1.4) will be extended to a differential Riccati equation with two point boundary conditions.

III. Dynamical Systems

The pair $[A, B]$, or the system (1.3), is said to be stabilizable if there exists a constant matrix L such that the system (1.3) is asymptotically stable with the control $u = Lx$. The following result is an extension of the well known theory on the algebraic matrix Riccati equation.

Theorem 3.1. $[A. B]$ is stabilizable if and only if for each C such that $[A, C]$ is observable, $F \geqslant 0$, and a scalar $T > 0$ there exists a $K(t) > 0$, $0 \leqslant t \leqslant T$, such that

$$-\dot{K} = A'K + KA - KBB'K + C'C \qquad (3.1)$$

$$K(T) - K(0) = F. \qquad (3.2)$$

Moreover, for a stabilizable pair $[A, B]$, $K(t) \geqslant K_s$ for a $F \geqslant 0$, and $K(t) = K_s$, $0 \leqslant t \leqslant T$, if and only if $F = 0$, where K_s is the solution of the algebraic Riccati matrix equation (1.4). Also the stabilizable system (1.3) is uniformly asymptotically stable with the periodic feedback gain control

$$u(t) = -B'\hat{K}(t)x(t), \qquad (3.3)$$

where $\hat{K}(t)$, $0 \leqslant t < \infty$, is defined as in (2.4) (Fig. 1 (a)).

The proof of Theorem 3.1 will be left until Theorem 3.2 is proven. The solution of (3.1)-(3.2) looks, at first glance, very difficult to solve since it is a two-point boundary value problem for a nonlinear differential equation. But it can be easily computed iteratively by solving linear equations as shown in Theorem 3.2.

Theorem 3.2. Assume $[A, B]$ is stabilizable and $[A, C]$ is observable. Let

$$-\dot{K}_{n+1} = (A-BB'K_n)'K_{n+1} + K_{n+1}(A-BB'K_n)$$
$$+ K_nBB'K_n + C'C \qquad (3.4)$$

$$K_{n+1}(T) - K_{n+1}(0) = F, \quad n=0,1,2,\ldots \qquad (3.5)$$

Then $K_n(t) \to K(t)$, $0 \leq t \leq T$, where $K(t)$ is the solution to (3.1)-(3.2) and $K_0(t)$ is chosen such that $A\text{-}BB'\hat{K}_0(t)$ is stable.

Proof: Assume that $A\text{-}BB'\hat{K}_n(t)$ is a stable matrix, $\alpha_1 I \leq K_n(t) \leq \alpha_2 I$, $0 \leq t \leq T$, for some $\alpha_1 > 0$ and $\alpha_2 > 0$, and $\Phi_n(t)$ is the state transition matrix of $A\text{-}BB'\hat{K}_n(t)$. Then $|\lambda_i(\Phi_n(t))| < 1$ [11]. Relation (3.5) is expressed as

$$\Phi_n'(T)K_{n+1}(0)\Phi_n(T) - K_{n+1}(0) = -\Phi_n'(T)F\Phi_n(T)$$

$$-\int_0^T \Phi_n'(t)(C'C+K_n(t)BB'K_n(t))\Phi_n(t)dt. \qquad (3.6)$$

$\dot{x}(t) = Ax(t) + Bu(t)$ where $u(t) = -B'\hat{K}_n'(t)x(t)$

consider a system (3.7)

and let $F'(t)F(t) = C'C + \hat{K}_n n(t)BB'\hat{K}_n(t)$. It is claimed that $[A\text{-}BB'\hat{K}_n(t), F(t)$ is observable. Consider a scalar value function

$$J(t_0, T) = \int_{t_0}^{T} (x'(t)C'Cx(t) + u'(t)u(t))dt$$

$$= x'(t_0)\int_{t_0}^{T}\Phi_n'(t, t_0)\,(C'C + \hat{K}_n(t)BB'\hat{K}_n(t))$$

$$\times\ \Phi_n(t, t_0)\,dt\ x\ (t_0).$$

If for a small interval $[t_1, t_2] \subset [0, T]$ $u(t)$ is not identically zero, then $J(t_1, t_2) > 0$ since $u(t)$ is piecewise continous. If for a small interval $[t_1, t_2] \subset [0, T]$ $u(t)$ is identically zero, then $J(t_1, t_2) = x'(t_1)\int_{t_1}^{t_2} e^{A(t-t_1)}$ $C'Ce^{A(t-t_1)}dtx(t_1) > 0$ since $[A, C]$ is observable. Thus $[A\text{-}BB'\hat{K}_n(t), F(t)]$ is observable. Since $\hat{K}_n(t)$ and $F(t)$ is periodic it is uniformly observable [14]. From this result the last term of (3.5) is negativedefinite and thus there exists a unique $K_{n+1}(0) > 0$. Since a scalar T in (3.4)-(3.5) is a finite value $K_{n+1}(t)$ is bounded from above. It will be shown that $K_{n+1} < K_n(t)$. From (3.4) follows that

$$-(\dot{K}_n-\dot{K}_{n+1}) = (A-BB'K_n)'(K_n-K_{n+1})+(K_n-K_{n+1})$$

$$(A-BB'K_n)+(K_n-K_{n-1})BB'(K_n-K_{n-1}) \qquad (3.8)$$

with a boundary condition $(K_n(T)-K_{n+1}(T))$ $-(K_n(0)-K_{n+1}(0)) = 0$. Let $K_{n,n+1} = K_n - K_{n+1}$. Then we obtain

$$\Phi_n'(T)K_{n,n+1}(0)\Phi_n(T) - K_{n,n+1}(0)$$

$$= -\int_0^T \Phi_n'(t)K_{n-1,n}(t)\Phi_n(t)dt, \qquad (3.9)$$

from which follows $K_{n,n+1}(0) = K_{n,n+1}(T) \geq 0$ and thus $K_{n,n+1}(t) \geq 0$ from (3.8)-(2.8). Now it will be shown that $A\text{-}BB'\hat{K}_{n+1}(t)$ is a stable matrix where $\hat{K}_{n+1}(t)$ is defined in (2.4). Consider a matrix $W_{n+1}(t)$ which satisfies

$$-\dot{W}_{n+1} = (A-BB'K_{n+1})'W_{n+1} + W_{n+1}(A-BB'K_{n+1})$$

$$+ K_{n+1}BB'K_{n+1} + C'C \qquad (3.10)$$

with an one-point boundary condition $W_{n+1}(0) = K_{n+1}(0)$. Then K_{n+1} and W_{n+1} satisfies Relation (3.8) with K_n and K_{n+1} replaced by K_{n+1} and W_{n+1} respectively. The boundary condition is $K_{n+1}(0) - W_{n+1}(0) = 0$. This relation and (2.7) implies that $K_{n+1}(t) - W_{n+1}(t) \leq 0$. Therefore $W_{n+1}(T) \geq K_{n+1}(T) \geq K_{n+1}(0) = W_{n+1}(0)$. Take $x'(t)\hat{W}_{n+1}(t)x(t)$ as a Lyapunov function for the matrix $A\text{-}BB'\hat{K}_{n+1}(t)$. Then this Lyapunov function satisfies all the conditions in Lemma 2.1. Since $\hat{W}_{n+1}(t) \geq K_{n+1}(0) > 0$ and is bounded from above because of a finite T in (3.10), it satisfies (2.2). Since $\hat{W}_{n+1}(t)$ is piecewise absolutely continuous on the interval $[t_s, t]$ and has jumps at jT, $j\epsilon\Omega = [j : jT\epsilon[t_s, t], t > t_s]$,

We have $\hat{W}_{n+1}(jT^+) - \hat{W}_{n+1}(jT^-) \geq 0$ and

$$V(t,x(t;x_s,t_s)) - V(t_s,x_s) = \int_{t_s}^{t}\dot{V}(\tau,x(\tau))d\tau$$

$$+\sum_{j\epsilon\Omega} x'(jT)[\hat{W}_{n+1}(jT^+)-\hat{W}_{n+1}(jT^-)]\,x(jT)$$

$$\leq -x'(t_s)\int_{t_s}^{t}\Phi_{n+1}'(t,t_s)(C'C+\hat{K}_{n+1}(t)BB'\hat{K}_{n+1}(t))\cdot\Phi_{n+1}(t,t_s)dtx(t_s) \qquad (3.11)$$

where $\Phi_{n+1}(t, t_0)$ is the state transition matrix of $A\text{-}BB'_{n+1}(t)$. Since it has been shown that $[A\text{-}BB'\hat{K}_{n+1}(t), F(t)]$ is uniformly observable where $F(t)$ is such that $F'(t)f(t) = C'C$ $\hat{K}_{n+1}(t)BB'\hat{K}_{n+1}(t)$, it satisfies the same relation (2.13) for some α_3 and some $d > 0$. Therefore the matrix $A\text{-}BB'\hat{K}_{n+1}(t)$ is a stable matrix. Consequently $K_{n+2}(t)$ is uniquely defined such that $0 < K_{n+2}(t) \leq K_{n+1}(t)$. Now we show that $K_{n+1}(t) \geqslant K_s$ where $K_s > 0$ is the positive definite solution of the algebraic Riccati equation (1.4). Let $K_{n+1,s} = K_{n+1}(t) - K_s$. Then we have

$$-\dot{K}_{n+1,s} = (A-BB'K_n)'K_{n+1,s} + K_{n+1,s}(A-BB'K_n)$$

$$+(K_n - K_s)BB'(K_n-K_s) \qquad (3.12)$$

with a boundary condition $K_{n+1,s}(T) - K_{n+1,s}(0) = F$, from which follows

$$\Phi_n'(T)K_{n+1,s}(0)\Phi_n(T) - K_{n+1,s}(0) = -\Phi_n'(T)F\Phi_n(T)$$

$$-\int_0^T \Phi_n'(t)K_{n,s}(t)BB'K_{n,s}(t)\Phi_n(t)dt. \qquad (3.13)$$

From (3.13) there exists $K_{n+1,s}(0) \geqslant 0$ and thus $K_{n+1,s}(T) \geqslant 0$. Therefore $K_{n+1,s}(t) \geqslant 0$ from the relation similar to (2.8). If [A, B] is stabilizable there exists $K_0(t)$ such that $A-BB'\hat{K}_0(t)$ is stable. Thus we have proved that there exist matrices K_i, $i = 1, 2, ..$ such that

$$K_1(t) \geqslant K_2(t) \geqslant \cdots\cdots\cdots \geqslant K_s. \qquad (3.14)$$

From a theorem on monotonic convergence of positive operators [15] there exists a positive definite matrix K(t) such that $\lim_{n\to\infty} K_n(t) = K(t) \geqslant K_s$. An equivalent form of (3.4),

$$K_{n+1}(t) = \Phi_n'(t)K_{n+1}(0)\Phi_n(t) - \int_0^t \Phi_n'(t)$$

$$(K_n(t)BB'K_n(t) + C'C)\Phi_n(t)dt,$$

approaches to

$$K(t) = \Phi'(t)\ K(0)\ \Phi(t) - \int_0^t \Phi(t)$$

$$(K(t)BB'K(t) + C'C)\ \Phi(t)dt \qquad (3.15)$$

from the dominated convergence theorem on integral. The relation (3.15) is an equivalent form of (3.1). This completes the proof.

It should be noted that the computation of (3.4)-(3.5) is easy since $K_{n+1}(0)$ can be obtained from the discrete algebraic Lyapunov matrix equation (3.6) which can be transformed to a linear equation. With this initial condition $K_{n+1}(0)$, the solution of (3.4) is easy since it is an one-point boundary value problem rather than a two-point boundary value problem. It is also noted that the constraint (3.5) can be relaxed such as follows:

$$K_{n+1}(T) - K_{n+1}(0) = F_{n+1} \qquad (3.16)$$

where F_i, $i = 1, 2, 3$, is chosen such that

$$F_1 \geqslant F_2 \geqslant F_3 \geqslant \cdot\ \cdot \geqslant F \qquad (3.17)$$

and $\lim_{n\to\infty} F_n = F$. The good part of the proof of Theorem 3.1 has been given in the proof of Theorem 3.2. The rest of it proceeds as follows:

Proof of Theorem 3.1 (Necessity) This is given in Theorem 3.2. (Sufficienty) Take $x'(t)\hat{K}(t)x(t) = V(t, x(t))$ as a Lyapunov function of the system (1.3) with a control (3.3). Then we have

$$V(t,x(t;x_s,t_s)) - V(t_s,x_s) = \int_{t_s}^t \dot{V}(t,x(t))dt$$

$$+\sum_{j\in\Omega} x'(jT)[\hat{K}(jT^+)-\hat{K}(jT^-)]\,x(jT)$$

$$\leqslant -x'(t_s)\int_{t_s}^t \Phi'(t,t_s)(C'C+\hat{K}(t)BB'\hat{K}(t))$$

$$\Phi(t,t_s)dt\ x(t_s) \qquad (3.18)$$

where $\Phi(t)$ is the state transition matrix of $A-BB'K(t)$. Since $[A-BB'K(t), F(t)]$ is uniformly observable where $F'(t)F(t) = C'C + K(t)BB'K(t)$, (3.18) satisfies the condition (2.3). Thus this Lyapunov function satisfies all conditions of Lemma 2.1. Therefore the control (3.3) is a stable control. Since linear time invariant system can not be stabilized with time-varying feedback control unless it is stabilizable with a constant feedback control, the system (1.3) is stabilizable. It is left to show that $K(t) = K_s$, $0 \leqslant t \leqslant T$, if and only if $F = 0$. In the case of $F = 0$, K_s can be a solution to (3.1)-(3.2) since it satisfies (3.1) and (3.2). The Property that K_s is the unique solution follows from the following result.

Theorem 3.3 If [A, B] is stabilizable and [A, C] is observable, then (3.1)-(3.2) has a unique positive definite matrix solution.

Proof: Let $K_1(t)$ and $K_2(t)$ are two different positive definite matrix solutions to (3.1)-(3.2). Then we have

$$-(\dot{K}_1-\dot{K}_2) = (K_1-K_2)(A-BB'K_1)$$

$$+(A-BB'K_2)'(K_1-K_2) \qquad (3.19)$$

$$(K_1(T)-K_2(T)) - (K_1(0)-K_2(0)) = 0. \qquad (3.20)$$

Let $K_{12}(t) = K_1(t) - K_2(t)$ and $\Phi_1(t)$ and $\Phi_2(t)$

are state transition matrices of $(A-BB'K_1(t))$ and $(A-BB'K_2(t))$ respectively. The above relations can be expressed as

$$\Phi_2'^{-1}(T)K_{12}(0)\Phi_1'^{-1}(T) - K_{12}(0) = 0$$

which is equivalent to

$$\Phi_2'^{-1}(T)K_{12}(0) + K_{12}(0)(-\Phi_1(T)) = 0.$$

Since $\lambda_i(\Phi'^{-1}(T)) + \lambda_j(-\Phi_1(T)) \neq 0$, a unique solution $K_{12}(0)$ is the zero solution. Thus $K_{12}(t) = 0$, $0 \leqslant t \leqslant T$, is the unique solution to (3.19), which implies $K_1(t) = K_2(t)$. This completes the proof.

The results in Theorems 3.1, 3.2 and 3.3 characterize not only the differential matrix Riccati equation with two-point boundary conditions but also the optimal control problem for a quadratic cost with a discrete state penalty. It can also be used as means for botaining linear periodic feedback gain controls. The computation of K(t) requires an initial guess of a stable control for the iterative method in Theorem 3.2. An easy method to obtain a linear stable periodic feedback gain control is suggested, which is easier to compute than the control (3.3) and can also be used as an initial guess for the iterative method.

Theorem 3.4 Assume that [A, B] is controllable. The system (1.3) is uniformly asymptotically stable with a control law

$$u(t) = -B'\hat{P}^{-1}(t)x(t) \tag{3.21}$$

where $P(t)$, $0 \leqslant t \leqslant T$, is obtained from

$$-\dot{P} = -AP - PA' - PC'CP + BB' \tag{3.22}$$

$$P(T+d) = 0$$

for any matrix C and $d > 0$ (Fig. 1 (b)).

Proof: Consider the adjoint system of (1.3) with the control law (3.21):

$$\dot{\hat{x}}(t) = -(A-BB'P^{-1}(t))'\hat{x}(t). \tag{3.23}$$

Take a Lyapunov function

$$V(t,\hat{x}(t)) = \hat{x}'(t)\hat{P}(t)\hat{x}(t). \tag{3.24}$$

It is well known that $P(t_1) \geqslant P(t_2)$ for $t_1 \leqslant t_2 \leqslant T+d$. Thus we have

$$\hat{P}(jT^+) - \hat{P}(jT^-) \geqslant 0, j = 1,2, \ldots \tag{3.25}$$

Since [A,B] is controllable and T is a finite value there exist $\alpha_4 > 0$ and $\alpha_5 > 0$ such that $\alpha_4 I \leqslant P(t) \leqslant \alpha_5 I$ and thus

$$\alpha_4|\hat{x}|^2 \leqslant V(t,\hat{x}) \leqslant \alpha_5|\hat{x}|^2. \tag{3.26}$$

From (3.24) and (3.25) we can have

$$\begin{aligned}
&V(t,\hat{x}(t;\hat{x}_s,t_s)) - V(t_s,\hat{x}_s) \\
&= \int_{t_s}^{t}\dot{V}(t,\hat{x}(t))dt + \sum_{j\in\Omega}\hat{x}(jT)[\hat{P}(jT^+)-\hat{P}(jT^-)]\hat{x}(jT) \\
&\geqslant \int_{t_s}^{t}\dot{V}(t,\hat{x}(t))dt \\
&= \int_{t_s}^{t}\dot{\hat{x}}'(t)\hat{P}(t)\hat{x}(t) + \hat{x}'(t)\hat{P}(t)\dot{\hat{x}}(t) + \hat{x}'(t)\dot{\hat{P}}(t)\hat{x}(t)dt \\
&= \int_{t_s}^{t}\hat{x}'(t)(-A\hat{P}(t)-\hat{P}(t)A'+2BB'+\dot{\hat{P}}(t))\hat{x}(t)dt \\
&= \int_{t_s}^{t}\hat{x}'(t)(\hat{P}(t)C'C\hat{P}(t) + BB')\hat{x}(t)dt \\
&\geqslant \hat{x}'_s\int_{t_s}^{t}\Phi_P(t_s,t)BB'\Phi_P'(t_s,t)dt\hat{x}_s
\end{aligned} \tag{3.27}$$

$$\geqslant \alpha_6|x_s|^2,\ t \geqslant t_s + \delta \tag{3.28}$$

where $\Phi_P(t, \tau)$ is the state transition matrix of $[A-BB'P^{-1}(t)]$ and δ and α_6 are some positive numbers. Since [A, B] is controllable $[A-BB'P^{-1}(t), B]$ is uniformly controllable [10] and its controllability matrix (3.27) is positive definite for some $\delta > 0$. From (3.26) and (3.28) the adjoint system (3.23) is exponentially increasing, i.e. $|\hat{x}(t)| \geqslant \alpha_7 e^{\beta t}$ for $\alpha_7 > 0$, $\beta > 0$ which in return implies that the system (1.3) with the control (3.21) is exponentially decreasing, i.e. $|x(t)| \leqslant \alpha_8 e^{-\gamma t}$ for $\alpha_7 > 0$ and $\gamma > 0$. This completes the proof.

From the special structure of a time-invariant system, the condition of Theorem 3.4 can be weakened as follows.

Proposition 3.1. If [A, B] is stabilizable, then the system is uniformly asymptotically stable with the following control law:

$$u(t) = -B'\hat{P}^{\dagger}(t)x(t)$$

where $\hat{P}^{\dagger}(t)$ is the generalized inverse of the matrix $\hat{P}(t)$ obtained from (3.22).

The proof of Proposition 3.1 can be carried out the same as in [6]. The control laws given in Theorem 3.4 and Proposition 3.1 not only provide easy means to obtain stable periodic feedback gain control laws but also can be used as initial stable control laws required for the iterative method given in Theorem 3.2. It will be interesting to see whether the control law (3.21) is optimal in some sense. From the way it is constructed the control law (3.21) is the optimal control law for the system (1.3) in the following sense [6]: Suppose the current time t belongs to an interval [(j−1)T, jT] for some j. The control u(t) at the current time t minimizes a quadratic cost

$$\int_{t}^{jT+d} (x'(s)C'Cx(s) + u'(s)u(s))ds \qquad (3.29)$$

subject to a state constraint

$$x(jT+d) = 0. \qquad (3.30)$$

It is noted that jT+d is a function of the current time t and jT+d changes when the current time t belongs to another interval. This terminal time will be called an intervalwise receding horizon in contrast with a usual pointwise receding horizon in [6, 7, 8].

V. Conclusion

This paper demonstrates that seemingly difficult two-point boundary value problems for differential Lyapunov and Riccati matrix equations have such good properties as those of algebraic matrix Lyapunov and Riccati equations. This work also characterizes the steady state optimal control with a quadratic cost with a discrete state penalty. An iterative computation method is suggested for two-point voundary Riccati equations whose computation, otherwise, is very difficult. For a linear stable periodic feedback gain control, an intervalwise receding horizon control law (3.21) is presented in this paper which can also be used as an initial guess of the iterative method. Numeircal examples show the validity of these methods. Robustness and asymptotic behavior of the optimal system with two-point boundary Riccati equations needs to be investigated along with standard stead state regulators.

Acknowledgement

This work was supported in part by the 1979 Science and Technology Research Fund of the Korean Ministry of Education.

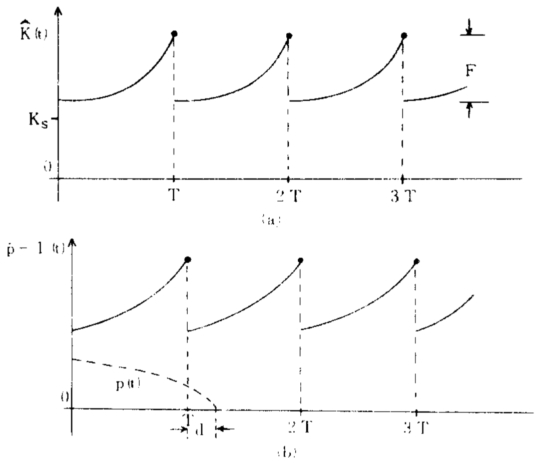

Fig. 1. Matrix K(t) in (3.3) and matrix $P^{-1}(t)$ in (3.21).

References

1. H.P. Geering, "Continuous-time optimal control theory for cost functionals including discrete state penalty terms", IEEE Trans. on Automatic Control, Vol. AC-21, December 1976.
2. R. E. Kalman, "When is a linear control system optimal," Trans. ASME, J. Basic Eng., Ser D, Vol. 86, pp. 51-60, 1964.
3. D. L. Kleinman, "On an iterative technique for Riccati equation computation," IEEE Trans. on Automatic Control, Vol. AC-13, No. 1, February 1968.

4. W. H. Kwon and A. E. Pearson, "A note on the Algebraic Matrix Riccati equation," IEEE Trans. on Automatic Control, Vol. AC-22, February 1977.

5. D. L. Kleinman, "An easy way to stabilize a linear constant system," IEEE Trans. on Automatic Control, Vol. Ac-15, December 1970.

6. W. H. Kwon and A. E. Pearson, "a modified quadratic cost problem and feedback stabilization of a linear system," IEEE Trans. on Automatic Control, Vol. AC-22, October 1977.

7. W. H. Kwon and A. E. Pearson, "On feedback stabilization of time-varying discrete linear systems," IEEE Trans. on Automatic Control, Vol. AC-23, June 1978.

8. W. H. Kwon and A. E. Pearson, "A double integral quadratic cost and tolerance of feedback nonlinearities," IEEE Trans. on Automatic Control, Vol. AC-24, June 1979.

9. R. E. Bellman and R. E. Kalaba, Quasilinearization and Nonlinear Boundary-value Problems, American Elsevier Publishing Co., N. Y., 1965.

10. B. D. O. anderson and J. B. Moore, "New results in linear system stability," SIAM J. Control, Vol. 7, August 1969.

11. Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, New York University, 1965.

12. J. K. Hale, Ordinary Differential Equations, Wiley-Interscience, N. Y., 1969.

13. S. Bittanti, G. Fronza, and G. Guardabassi, "Periodic control: A frequency Domain approach," IEEE Trans. on Automatic Control, Vol. AC-18, February 1973.

14. L. M. Silverman and B. D. O. Anderson, "Controllability, observability and stability of linear systems", SIAM J. Control, Vol. 6, 1968.

15. G. Helmberg, Introduction to Spectral Theory in Hilbert Space, John Wiley & Sons, Inc., New York, 1969.