J. of The Korean Operations
Research Society
Vol. 4, No. 1, April 1979.

Economic Production Quantity with Expontial Deterioration

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Abstract

Production lot sizing problem for a system with exponentially decaying inventory is considered. From the exact cost function developed under conditions of constant demand and no shortages permitted, an approximate optimal solution is derived.

The formula is compared with those of the exact solution obtained from numerical procedure and other existing approximate solution.

Finally some notable properties of the formula are investigated and shown to be consistent.

1. Introduction

Recently several inventory models have been considered in which inventory is depleted not only by physical depletion but also by decay.

Since Ghare and Schrader [3] derived a revised form of the economic ordering quantity in which the time elapsed to deterioration has exponential distribution, Covert and Philip [2] extended this model to the case of weibull distribution.

Shah [7] has formulated an economic ordering quantity model in which the time to deterioration is expressed in a general probabilistic distribution. Cohen [1] considered the problem of simultaneously setting price and production levels for an exponentially decaying product.

Misra [5] has developed a production lot size model for a system with deteriorating inventory. We find Misra's model for the case of weibull deterioration to have a wrong basic assumption in which he assumes the deterioration rate function is simply the weibull instantaneous rate function, $Z(t)=\alpha\beta t^{\beta-1}$. He ignores the fact that after the start of production the stockpile consists of items of different ages. For exponential deterioration problem, the complexity of the problem mentioned above does not occur because the deterioration rate function is independent of the time elapsed. In this case Misra has developed a rough formula for optimal solution. In this paper production lot size model with expenentially decaying inventory is considered and an improved approximate solution is developed comparing with the Misra's result.

2. Development of the Model

Following are the list of assumptions we make to develop the model

1) Demand rate is known and constant

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2) Production rate, governing supply is finite and constant
3) Units are available for satisfying demand after their production
4) A deteriorated unit is not repaired or replaced
5) The unit production cost, C is also considered to account for the deterioration cost, and all the cost coefficients are constant
6) The production rate is greater than the demand rate
7) Shortages are not allowed
8) The time elapsed to deterioration follows exponential distribution with parameter α.

NOTATION

The notation used in this paper is as follows;

ϕ = Production rate in number of units/year
λ = Demand rate in number of units/year
C = Unit production cost
C_1 = Inventory carrying cost/unit/unit time.
C_3 = Set-up cost per cycle
I_t = The inventory level at time t
I_0 = Maximum inventory level within a cycle
T = Cycle time
T_1 = Production time per cycle
T_2 = Time without production in a cycle
i.e., $T_2 = T - T_1$
TC = Total cost per unit time
T_1^* = Optimal value of T_1
T_2^* = Optimal value of T_2
T_{1C}^* = Optimal value of T_1 for conventional production lot size model
T_{2C}^* = Optimal value of T_2 for conventional production lot size model

Figure-1 shows an inventory cycle for a finite production rate. The inventory level is zero both at the beginning and end of the cycle. During time interval $[0, T_1]$ the inventory level increases due to production and decreases after production stops at time T_1.

The depletion, dI, during the infinitesimal time, dt following t, is a function of the deterioration, the rate of usage, the production rate ϕ and the remaining inventory. Thus

$$-dI_t = I_t\alpha dt + \lambda dt - \phi dt \quad \text{for } 0 \leq t \leq T_1 \tag{1}$$

and

$$-dI_t = I_t\alpha dt + \lambda dt \quad \text{for } T_1 \leq t \leq T \tag{2}$$

Equation (1) and (2) can be rewritten as

$$\frac{dI_t}{dt} + I_t\alpha = \phi - \lambda, \quad 0 \leq t \leq T_1 \tag{3}$$

and

$$\frac{dI_t}{dt} + I_t\alpha = -\lambda, \quad T_1 \leq t \leq T. \tag{4}$$

It follows

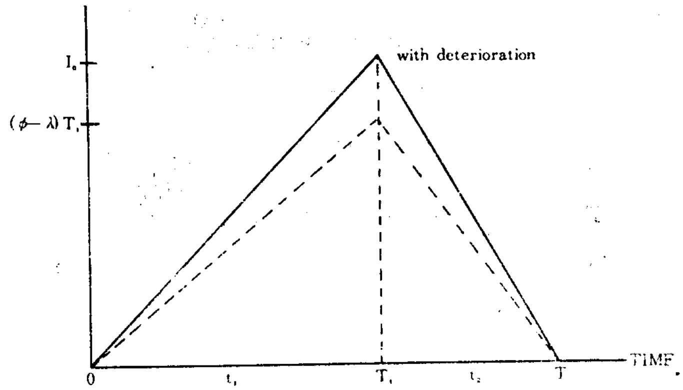

Figure 1 Behaviour of on hand inventory

$$I_t=\frac{\int_0^t(\phi-\lambda)\ \exp\left(\int\alpha dt\right)dt+K_1}{\exp\left(\int_0^t\alpha dt\right)},0\le t\le T_1 \tag{5}$$

and

$$I_t=\frac{\int_{T1}^t(-\lambda)\exp\left(\int\alpha dt\right)dt+K_2}{\exp\left(\int_{T1}^t\alpha dt\right)},\ T_1\le t\le T \tag{6}$$

Using the boundary conditions that at $t=0$, $I_t=0$ and at $t=T_1, I_t=I_0$, $K_1=0$ and $K_2=I_0$, let $t_1=t$ for $0\le t\le T_1$

and

$t_2=t-T_1$ for $T_1\le t\le T$.

Then

$$I_{t1}=\frac{\int_0^{t_1}(\phi-\lambda)\exp(\alpha t)\ dt}{\exp(\alpha t_1)}=\frac{\phi-\lambda}{\alpha}[1-\exp(-\alpha t_1)] \tag{7}$$

$$I_{t2}=\frac{\int_0^{t_2}(-\lambda)\exp(\alpha t)dt+I_0}{\exp(\alpha t_2)}=\frac{-\frac{\lambda}{\alpha}[\exp(\alpha t_2)-1]+I_0}{\exp(\alpha t_2)} \tag{8}$$

From the fact that at $t_1, I_{t1}=I_0$ and at $t_2=T_2$, $I_{t2}=0$, it follows that

$$I_0=\frac{\lambda}{\alpha}[\exp(\alpha T_2)-1]=\frac{\phi-\lambda}{\alpha}[1-\exp(-\alpha T_1)] \tag{9}$$

and

$$I_{t2}=\frac{\lambda[\exp(\alpha T_2)-\exp(\alpha t_2)]}{\alpha\exp(\alpha t_2)} \tag{10}$$

From (9), it follows that

$$T_2=\frac{1}{\alpha}ln\left[\frac{\phi}{\lambda}-\frac{\phi-\lambda}{\lambda}\exp(-\alpha T_1)\right], \tag{11}$$

$$\int_0^{T_2}I_{t2}dt=\frac{\lambda}{\alpha}\left[-\frac{1}{\alpha}-T_2+\frac{1}{\alpha}\exp(\alpha T_2)\right] \tag{12}$$

and

$$\int_0^{T_1}I_{t1}dt=\frac{\phi-\lambda}{\alpha}\left[T_1+\frac{1}{\alpha}\exp(-\alpha T_1)-\frac{1}{\alpha}\right]. \tag{13}$$

We can express equation (12) and (13) in terms of T_1 using equation (11).

$$\int_0^{T_1} I_{t1}dt + \int_0^{T_2} I_{t2}dt = \frac{\phi-\lambda}{\alpha}T_1 - \frac{\lambda}{\alpha^2} ln\left[\frac{\phi}{\lambda} - \frac{\phi-\lambda}{\lambda}\exp(-\alpha T_1)\right] \quad (14)$$

Thus the total inventory cost per unit time becomes

$$TC = \frac{C_3 + C\phi T_1 + C_1\frac{\phi-\lambda}{\alpha}T_1 - \frac{C_1\lambda}{\alpha^2} ln\left[\frac{\phi}{\lambda} - \frac{\phi-\lambda}{\lambda}\exp(-\alpha T_1)\right]}{T_1 + \frac{1}{\alpha} ln\left[\frac{\phi}{\lambda} - \frac{\phi-\lambda}{\lambda}\exp(-\alpha T_1)\right]} \quad (15)$$

Here we notice that the above cost function (15) is not convex because

$$\lim_{T_1\to\infty} TC = C\phi + C_1\frac{\phi-\lambda}{\alpha}$$

Using Rosen's gradient projection method [4], optimal solution which minimizes the total cost per unit time can be obtained. To derive approximate solution, those terms of degree higher than or equal to 2 in α are neglected in Taylor's expansion of the function (15) assuming $\alpha \ll 1$.

Then

$$TC = \frac{C_3 + C\phi T_1 + \frac{C_1}{2}\frac{\phi(\phi-\lambda)}{\lambda}T_1^2 - \frac{C_1\alpha}{3}\left(\frac{\phi-\lambda}{\lambda}\right)^2\phi T_1^3 - \frac{C_1\alpha}{6}\left(\frac{\phi-\lambda}{\lambda}\right)\phi T_1^3}{\frac{\phi}{\lambda}T_1 - \frac{\alpha}{2}\left(\frac{\phi-\lambda}{\lambda}\right)\frac{\phi}{\lambda}T_1^2} \quad (16)$$

Differentiating with respect to T_1 and set equal to zero, we obtain

$$-\frac{1}{3}C_1\alpha(2\phi-\lambda)\frac{\phi}{\lambda}T_1^3 + \frac{1}{2}\phi(C\alpha+C_1)T_1^2 + C_3\alpha T_1 - \frac{C_3\lambda}{\phi-\lambda} = 0 \quad (17)$$

Assuming $\alpha T_1 \ll 1$, above equation can be solved for the optimum T_1^*.

This gives

$$T_1^* = \sqrt{\frac{2C_3\lambda}{(\phi-\lambda)\phi(C\alpha+C_1)}} = \sqrt{\frac{2C_3\lambda}{(\phi-\lambda)C_1\phi}}\sqrt{\frac{1}{\frac{C}{C_1}\alpha+1}} \quad (18)$$

Misra [5] developed a formula for T_1 and he gives

$$T_1^0 = \sqrt{\frac{2C_3\lambda}{(\phi-\lambda)(C\alpha\lambda+C_1\phi)}}$$

The same approximation procedures are applied to the total cost per unit time which is expressed in terms of T_2 and by setting $\frac{\partial TC}{\partial T_2} = 0$, following equation is obtained

$$T_2^* = \sqrt{\frac{2C_3(\phi-\lambda)}{\phi\lambda(C\alpha+C_1)}} = \sqrt{\frac{2C_3(\phi-\lambda)}{C_1\phi\lambda}}\sqrt{\frac{1}{\frac{C}{C_1}\alpha+1}} \quad (19)$$

Notice that as α approaches 0, T_1^* and T_2^* approaches to the optimal solution of production lot size problem without deterioration, that is

$$T_{1c}^* = \sqrt{\frac{2C_3\lambda}{(\phi-\lambda)C_1\phi}} \text{ and } T_{2c}^* = \sqrt{\frac{2C_3(\phi-\lambda)}{C_1\phi\lambda}}$$

So we find that an approximate solution can be found simply by mutiplying $\sqrt{\frac{1}{C\alpha/C_1+1}}$ to the solution of the ordinary production lot size problem.

Furthermore, we can consider the deteriorating cost as a kind of inventory holding cost. If we substitute $C\alpha+C_1$ for C_1 in the formulas, T_{1C}^* and T_{2C*}, we can obtain the formulas, T_1^* and T_2^*.

Also notice that T_1^* and T_2^* give standard result if the production rate is infinite. Ghare and

Schrader (1963) [3] have solved the above problem and obtained the following equation for optimal cycle time T:

$$\frac{C\lambda\alpha}{2}+\frac{C_1\lambda}{2}+\frac{C_1\lambda\alpha T}{2}-\frac{C_3}{T^2}=0 \qquad (20)$$

Recalling that αT is assumed to be quite small in the dervation of (18) and (19), from (20) $T^*=\sqrt{\frac{2C_3/C_1\lambda}{1+\alpha C/C_1}}$: which is the expression for T_2^* given by equation (19) after substituting $\phi=\infty$.

Thus the formula established in this paper is consistent.

3. Numerical Example

For the comparison purpose, we use the same numerical example appearing on Misra's paper. The values of various variables are as follows

$\lambda=2500$ unit/year

$\phi=7500$ unit/year

$C_1=$ \$0.60/unit/year

$C=$ \$3.00/unit

$C_3=$ \$50.00/order

Table-1 shows the value of T_1 and total cost per unit time derived from three different methods.

T_1^* appears to be better than T_1^0 and quite close to the optimal solution obtained by numerical method.

Table 1. optimal value of T_1 and Total Cost Per Unit Time.

α	0.001	0.051	0.101	0.151	0.201	0.251	0.301	0.351	0.401	0.451
Misra's Formula	0.1053 7817.0	0.1011 7855.4	0.0975 7891.3	0.0942 7925.1	0.0912 7957.1	0.0885 7987.5	0.0860 8016.5	0.0837 8044.2	0.0816 8070.9	0.0796 8096.5
New Formula	0.1051 7817.0	0.0941 7854.5	0.0859 7888.5	0.0796 7919.8	0.0744 7948.9	0.0702 7976.3	0.0666 8002.2	0.0635 8026.8	0.0608 8050.4	0.0584 8073.0
Exact Solution	0.1052 7817.0	0.0944 7854.5	0.0864 7888.5	0.0802 7919.8	0.0752 7948.9	0.0711 7976.2	0.0675 8002.1	0.0645 8026.8	0.0618 8050.3	0.0594 8073.0

4. Conclusion

An economic production quantity model has been considered in which items follow expenential deterioration and an approximate solulion is derived which is shown to be better than the Misra's result.

Cohen [1] has developed a joint pricing and ordering model in which a continuous review, deterministic demand and instantaneous supply are assumed. Extension of this problem to the production lot size model can be considered.

References

1. Cohen, M.A., "Joint Pricing and Ordering Policy for Exponentially Decaying Inventory with Known

Demand," Naval Research Quarterly, June 1977, Vol. 24, No. 2.

2. Covert, R.P. & Philip, G.C., "An EOO Model for Items with Weibull Distribution Deterioration," AIIE Transactions, 5, 4, 1973.
3. Ghare, P.M. & Schader, G.F., "A Model for Exponentially Decaying Inventory," Jr. of Ind. Engg., XIV, 243, 1963.
4. Kuester, J.L. and Mize, J.H., "Optimization Techniques with Fortran." McGRAW-Hill, New York, 1973.
5. Misra, R.B., "Optimum Production Lot-size Model for a System with Deteriorating Inventory," Int. Jr. of Prod. Rer., 13, 495-505, 1975.
6. Philib, G.C., "A Generalized EOQ Model for Items with Weibull Djstribution Deterioration'" AIIE Transactions, 6, 2. 159-162, 1974.
7. Shah, Y.K., "An Order-Level Lot-Size Inventory Model for Deteriorating Items," AIIE Transaction, 9. 1, 1977.