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Allocation of Aircraft under Uncertain Demand by Wets' Approach to Stochastic Programs with Simple Recourse

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Abstract

The application of optimization techniques to the planning of industrial, economic, administrative and military activities with random technological coefficients has been extensively studied in the literature. Stochastic (linear) programs with simple recourse essentially model the allocation of scarce resources under uncertainty with linear penalties associated with shortages or surplus. This work on a problem with a discrete random resource vector, "The allocation of aircraft under uncertain demand" given in (1), is easily and efficiently handled by the application of the recently developed Wets' algorithm (8) for solving stochastic programs with simple recourse, which approves that such class of stochastic problems can be solved with the same efficiency as solving linear programs of the same size. It is known that the algorithm is also applicable to stochastic programs with continuous random demands for their approximate solutions.

1. Introduction

The stochactic programming is an optimization technique which is applied to the planning of industrial, economic, administrative and military activities. These activities have the following common characteristics; a limited quantity of capital equipment or final product must be allocated among a number of final-use activities, where the corresponding demand level, and hence the payoff, is uncertain. Furthermore, once the allocation is made, minimum lead time or differences in form of the final products may not get the reallocation economically feasible.

In the formulation of applied problems, we assume more realistically that the distribution of the technological coefficients is known, whereas in the classical situation these coefficients are assumed to be completely given. Because of this stochastic nature of the formulation, we have some conceptual freedom in setting up the form of objective functions, treatment of constraints, and so on.

Let's consider a problem of minimizing CX, subject to $AX=b$, $X\geq 0$ and possibly to some other constraints as well, where C, A and b are fixed matrices of size $1\times n$, $m\times n$ and $m\times 1$, respectively.

Assume that b is not known, but only its distribution function. Then, the discrepancy between AX and b will itself be a random variable, whose distribution function depends on X. Therefore,

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we might have to deal with minimizing the sum of CX and the expected value of the associated potential penalties for any discrepancy as follows;

$$\begin{aligned} &\text{Minimize } Z(X)=CX+E\{\min qY\} \\ &\text{subject to } \quad AX+DY=b \\ &\qquad\qquad\qquad X,\ Y\geqslant 0, \end{aligned} \tag{1}$$

where E is an expectation operator, and q and D are matrices of size $1\times n_1$ and $m\times n_1$ respectively.

This programing means that X has to be found "here and now," before the actual values of b become known and then a so-called "recourse" Y must be found from the following second-stage program;

$$\begin{aligned} &\text{Minimize } qY \\ &\text{subject to } \quad DY=b-AX, \\ &\qquad\qquad\qquad Y\geqslant 0 \end{aligned} \tag{2}$$

where b and X are known.

In the literature, a stochastic program with recourse (random right-hand sides) is defined as follows;

$$\begin{aligned} &\text{Find inf } Z(X)=CX+E\xi\{\inf qY\} \\ &\text{subject to } \quad AX=b \\ &\qquad\qquad\qquad TX+WY=\xi \\ &\qquad\qquad\qquad X,Y\geq 0 \text{ and } \xi\in(\Omega,\ F,\ \mathrm{F}) \end{aligned} \tag{3}$$

where T and W are fixed matrices of dimension $m_1\times n$ and $m_1\times n$, respectively, and ξ is a random variable defined on the m_1-dimensional probability space $(\Omega,\ F,\ \mathrm{F})$

Denote by pos W the positive hull of the set determined by the points corresponding to the columns of a matrix W; i.e., pos $W=\text{pos}\ \{W^1,\cdots,W^i\}$. We speak of a stochastic program with "complete recourse" if pos $W=R^{m_1}$ and with "simple recourse" if $W=(I,-I)$, where I is the identity matrix of order m.

Over a decade of period, a lot of authors have studied on the development of algorithmic procedures for solving stochastic program problems. Among them, Wets [8] has recently developed an algorithm for solving stochastic programs with simple recourse, which is generated by a linear programming problem with stochastic coefficients and a specific loss function (piecewise linear) associated with the discrepancies observed between the output and the resource vector. The algorithm is constructed mainly for the cases of discrete random resource vectors, but its extensive use for random variables with a continuous distribution is also possible. It is approved by this work on the allocation of aircraft under uncertain demand given in [1] that the above-mentioned type of stochastic programs with simple recourse can be solved with the same efficiency as solving linear programs of the same size.

2. The Working Form of Stochastic Programs with Simple Recourse.

The program in expression (3) models the allocation of scarce resources under uncertainty with linear penalties (q^+Y^+) associated with shortages and (q^-Y^-) with surplus.

The problem in (3) can be reduced into tho following deterministic equivalent program;

$$\begin{aligned}
&\text{Find inf } Z(X) = CX + \Phi(\mathscr{T}_X) \\
&\text{subject to} \quad AX = b \\
&\qquad\qquad\quad X \geqq 0 \qquad\qquad (4) \\
&\text{where } \mathscr{T}_X = TX \text{ and} \\
&\qquad \Phi(\mathscr{T}_X) = E_\xi [\inf_Y \{qY \mid TX + WY = \xi,\ Y \geqslant 0\}]
\end{aligned}$$

It is not hard to show that in general the deterministic equivalent program is convex separable. According to [8], with the penalty coefficients satisfying the relation of $q_i = q^+_i + q^-_i > 0$ (for $i = 1, \cdots n_1$) it is sure that the recourse (second-stage) problems

$$\begin{aligned}
&\text{Minimize } \Psi_j = q^+_j Y^+ + q_j^- Y^- \\
&\text{subject to } Y_j^+ - Y_j^- = \xi_j - T_j \qquad\qquad (5) \\
&\qquad\qquad Y^+ \geqslant 0, Y_j^- \geqslant 0
\end{aligned}$$

where $\mathscr{T}_j$ is the j^{th} element of TX,

are not only feasible for all $\mathscr{T}_j \in R$ whatever the value of $\xi_j \in \Omega_i$ is, but are also solvable (finite and the infimum is actually attained by a feasible solution). Furthermore. there is an optimal solution to the problem with $Y_j^+ Y_j^- = 0$ which requires the utilization of at most one of the recourse activities.

In general, the feasible region of a stochastic program is obtained as the intersection of the set $K_2 = \{X \mid \Phi(\mathscr{T}_X) < \infty\}$ determined by the induced constraints and the set $K_1 = \{X \mid AX = b, X \geqslant 0\}$ determined by the fixed constraints.

Denote by $\leqslant C$ a partial cone ordering induced by a closed convex cone $C \subset R^{m_1}$. Given a cone ordering $\leqslant C$ and $S \subseteq R^{m_1}$, a point α_C is a lower bound of S with respect to the ordering induced by C if $\alpha_C \leqslant CU$ for all U in S. Furthermcre, it is a proper lower bound if $\alpha_C \in S$. Then, the following theorems are stated (see [8] for proof);

Theorem 1.

a) Assume C is a convex cone contained in pos W and α_C is a proper lower bound of Ω with respect to $\leqslant C$. Then the stochastic program is feasible iff the linear system $AX = b$, $TX + WY = \alpha_C$, $X \geqslant 0$, $Y \geqslant 0$ has a solution.

b) Assume $d \in \Omega$. Then the stochastic program (4) is bounded iff the linear program

$$\begin{aligned}
&\text{Minimize } Z(X) = CX + qY \\
&\text{subject to } AX = b \\
&\qquad\qquad TX + WY = d \\
&\qquad\qquad X \geqslant 0,\ Y \geqslant 0
\end{aligned}$$

is bounded.

c) Assume Ω is compact and there is X^1 such that $X + \lambda X^1 \in K_2$ for $\lambda \geqslant 0$ and some X in K_2. Then, there exists $\bar{\lambda} \geqslant 0$ such that $\Phi(\mathscr{T}_{X+TX^1})$ is linear in λ for $\lambda \geqslant \bar{\lambda}$. Furthermore, it is solvable if the stochastic program is bounded.

Now, for programs with discrete random variables the canonical equivalent (deterministic) program formulations [8] can be obtained from the program sets of (4) and (5);

$$\text{Find inf } Z(X) = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m_1} \sum_{l=0}^{k_i} p_{il} y_{il} \qquad (6)$$

subject to $\sum_{j=1}^{n} a_{ij}x_j = b_i \ (i=1, \ldots, m)$

$$\sum_{j=1}^{n} t_{ij}x_j - \sum_{l=0}^{ki} y_{il} = \alpha_i, \ (i=1, \ldots, m_1)$$

$$y_{il} + s_{il} = d_{il} \ (i=1, \ldots, m_1 \text{ and } l=0, \cdots k_i)$$

$$x_j \geqslant 0, \ y_{il} \geqslant 0, \ s_{il} \geqslant 0$$

where s_{il} = slack variables,

$$P_{il} = \sum_{j=1}^{l} p_{ij} = p_r\{\underline{\xi_i} \leq \xi_{il}\} = F_i\{\xi_{il}\}, \text{ with } p_{ij} = p_r\{\underline{\xi_i} = \xi_{ij}\},$$

$$p_{il} = -q_i^+ (1 - F_{il}) + q_i^- F_{il},$$

$$\alpha_i \leq \xi_{i1} \leq \xi_{i2} \leq \cdots \leq \xi_{ki} \leq \beta_i$$

$$d_{il} = \xi_{il+1} - \xi_{il}, \text{ with } d_{i0} = \xi_{i1} - \alpha_i \text{ and } d_{ik_i} = \beta_i - \xi_{ik_i}$$

In addition, there is the following properties in (6);

a) $y_{il} < d_{il}$ implies $y_{ij} = 0$ and $s_{ij} = d_{ij}, (j = l+1, \cdots k_i)$

b) $s_{il} < d_{il}$ implies $s_{ij} = 0$ and $y_{ij} = d_{ij}, (j = 0, \cdots, l-1)$

c) For solvable problems, at least one of the variables in each pair (y_{il}, s_{il}) is basic.

Let's define $U(i)$ and $B(i)$ as follows ;

$$U(i) = \{j \mid y_{ij} \text{ and } s_{ij} \text{ are both basic}\}$$
$$= \phi, \text{ otherwise}$$
$$B(i) = \{j \mid y_{ij} \text{ is basic and } j \notin U(i)\}$$

When the set of nonbasic $\{y_{ij}\}$ and $B(i)$ are known, the Wets' working form is formulated as follows;

$$\text{Find inf } Z(X) = \sum_{j=1}^{n} c_j x_j + \sum_{\{i \mid U(i) \neq \phi\}} p_{il} u_i$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i=1, \ldots, m) \tag{7}$$

$$\sum_{j=1}^{n} t_{ij}x_j = \tilde{\alpha}_i, \ (i=1, \cdots, m_1 \text{ and } U(i) = \phi)$$

$$\sum_{j=1}^{n} t_{ij}x_j - U_i = \tilde{\alpha}_i, \ (i=1, \cdots, m_1 \text{ and } U(i) \neq \phi)$$

where $x_j \geqslant 0 \ 0 \leq u_i \leq d_{il}$

$l \in U(i)$ and $\tilde{\alpha}_i = \alpha_i + \sum_{j \in B(i)} d_{ij}$

3. Application

The Wets' algorithm developed based on the expression (6) is applied to a problem in [1] which is to allocate several types of aircrafts over a number of routes, where only the distribution of the random monthly demands for service to each route is assumed to be known. The objective is to minimize the sum of the cost of performing the transportation plus the expected value of the revenue lost through the failure to serve all the traffic occurred. The informations on the problem are in Tables 1, 2 and 3.

The passenger-carrying capabilities $\{t_{ij}\}$ in hundreds, the cost $\{C_{ij}; i=1, \cdots, 4\}$ in thousands of dollars per month per aircraft of type i assigned to the route j and the revenue losses $\{C_{5j}\}$ in thousands of dollars per hundred passengers not carried (or costs from passengers turned away) are given in Table 2, where $t_{5j}=1$ is set to make easier to state the passenger-balance.

Table 1. The Number cf Available Aircraft Fleets

TYPE	DESCRIPTION	NUMBER AVAILABLE
A	Postwar 4-engine	10
B	Postwar 2-engine	19
C	Prewar 2-engine	25
D	Prewar 4-engine	15

Table 2. Passenger-Carrying Capabilities and Costs

Type of Aircraft	Route				
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$ (A)	$t_{11}=16$ $c_{11}=18$	$t_{12}=15$ $c_{12}=21$	$t_{13}=28$ $c_{13}=18$	$t_{14}=23$ $c_{14}=16$	$t_{15}=81$ $c_{15}=10$
$i=2$ (B)	0	$t_{22}=10$ $c_{22}=15$	$t_{23}=14$ $c_{23}=16$	$t_{24}=15$ $c_{24}=14$	$t_{25}=57$ $c_{25}=9$
$i=3$ (C)	0	$t_{32}=5$ $c_{32}=10$	0	$t_{34}=7$ $c_{34}=9$	$t_{35}=29$ $c_{35}=6$
$i=4$ (D)	$t_{41}=9$ $c_{41}=17$	$t_{42}=11$ $c_{42}=16$	$t_{43}=22$ $c_{43}=17$	$t_{44}=17$ $c_{44}=15$	$t_{45}=55$ $c_{45}=10$
Per hundred passengers not carried (losses)					
$i=5$ (E)	$t_{51}=1$ $c_{51}=13$	$t_{52}=1$ $c_{52}=13$	$t_{53}=1$ $c_{53}=7$	$t_{54}=1$ $c_{54}=7$	$t_{55}=1$ $c_{55}=1$

Table 3 shows the passenger demands $\{\xi_{hj}\}$ in hundreds to route j and the corresponding frequency distridutions $\{p_{hj}\}$ of the traffic demands on each route.

Table 3. Passenger Demand Distribution

Level of Demand	Route				
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$h=1$	$\xi_{11}=200$ $p_{11}=0.2$	$\xi_{12}=50$ $p_{12}=0.3$	$\xi_{12}=140$ $p_{13}=0.1$	$\xi_{14}=10$ $p_{14}=0.2$	$\xi_{15}=580$ $p_{15}=0.1$
$h=2$	$\xi_{21}=220$ $p_{21}=0.05$	$\xi_{22}=150$ $p_{22}=0.7$	$\xi_{23}=160$ $p_{23}=0.2$	$\xi_{24}=50$ $p_{24}=0.2$	$\xi_{25}=600$ $p_{25}=0.8$
$h=3$	$\xi_{31}=250$ $p_{31}=0.35$	0	$\xi_{33}=180$ $p_{33}=0.4$	$\xi_{34}=80$ $p_{34}=0.3$	$\xi_{35}=620$ $p_{35}=0.1$
$h=4$	$\xi_{41}=270$ $p_{41}=0.2$	0	$\xi_{43}=200$ $p_{43}=0.2$	$\xi_{44}=100$ $p_{44}=0.2$	0
$h=5$	$\xi_{51}=300$ $p_{51}=0.2$	0	$\xi_{53}=220$ $p_{53}=0.1$	$\xi_{54}=340$ $p_{54}=0.1$	0

In view of Theorem 1 and expression (6), this problem can be solved with the following coefficient set-up;

xij = the number of aircraft type i assigned to route j ($i=1,2,\ldots,5$ and $j=1,\ldots,5$) $=x_k$ where $k=(i-1)\times 4+j$

Cij = the cost in thousand dollars per month per aircraft type i to route $j=C_k$

$b=\{b_i\}$ = total number of aircraft type $i=(10, 19, 25, 15)$

$q^+=\{q_j^+\}$ = shortage cost not carrying waiting passengers to route $j=(13, 13, 7, 7, 1)$

$q^-=\{q^-_j\}$ = cost of aircraft overassignment to route $j=(0, 0, 0, 0, 0)$

$$A_{4\times 20}=\begin{pmatrix} \lambda & \phi & \phi & \phi \\ \phi & \lambda & \phi & \phi \\ \phi & \phi & \lambda & \phi \\ \phi & \phi & \phi & \lambda \end{pmatrix}, \text{ where } \begin{array}{l} \lambda=(1,\ 1,\ 1,\ 1,\ 1) \\ \phi=(0,\ 0,\ 0,\ 0,\ 0) \end{array}$$

$$T_{5\times 20}=\{t_{jk}\},\ \text{for } k=(i-1)\times 4+j$$

$$=\begin{pmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 & 0 & 0 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22 & 0 & 0 \\ 0 & 0 & 0 & 23 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 81 & 0 & 0 & 0 & 0 & 57 & 0 & 0 & 0 & 0 & 29 & 0 & 0 & 0 & 0 & 55 \end{pmatrix}$$

$$\left.\begin{array}{l} \alpha_i=0 \\ \beta_j=700 \end{array}\right\}(i=1,\ 2,\ldots,5).$$

4. Conclusion

This work shows that the final soultion is easily and very efficiently obtained by the algorithmic appreach of the ordinary simplex method as compared with the work in [1]. Since this algorithm can also be applied to stochastic programs with continuous random demands for their approximate solutions, the wide range of its applications will be certain whenever the corresponding distributions are found.

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