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# Optimal Dynamic Operating Policies for a Tandem Queueing Service System

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## ABSTRACT

This paper considers the problem of determining an optimal dynamic operating policy for a two-stage tandem queueing service system in which the service facilities (or stages) can be operated at more than one service rate. At each period of the system's operation, the system manager must specify which of the available service rates is to be employed at each stage. The cost structure includes an operating cost for running each stage and a service facility profit earned when a service completion occurs at Stage 2. We assume that the system has a finite waiting capacity in front of each station and each customer requires two services which must be done in sequence, that is, customers must pass through Stage 1 and Stage 2 in that order. Processing must be in the order of arrival at each station.

The objective is to minimize the total discounted expected cost in a two-stage tandem queueing service system, which we formulate as a Discrete-Time Markov Decision Process. We present analytical and numerical results that specify the form of the optimal dynamic operating policy for a two-stage tandem queueing service system.

## DESCRIPTION OF THE PROBLEM

This study considers the problem of determining an optimal dynamic operating policy for a two-stage tandem queueing service system in which the service facilities (or stages) can be operated at more than one service rates. At each period of the system's operation, the system manager must specify which of the available service rates is to be employed at each stage. The cost structure includes an operating cost for running each stage and a service facility profit earned when a service completion occurs at Stage 2.

We assume that the system has a finite waiting capacity in front of each stage and each customer requires two services which must be done in sequence, that is, customers must pass through Stage 1 and Stage 2 in that order. Processing must be in the order of arrival at each station. Let "a" be a certain action which the manager chooses. We assume that if there is a customer in Stage 1 and Queue 2 is not full at the beginning of the period, then there is a probability $\alpha(a)$, which is a function of action a, of completing the service at Stage 1 in that period. If there is a customer in Stage 2 at the beginning of the period, then there is a probability $\mu(a)$ o completing the service at Stage 2 in that period. Also there is a probab-

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ility of an arrival in the period (t, t+△t) if Queue 1 is not full. Finally, we assume that the system operates in discrete time with a small interval between observations and at most, one event occurs in a period (t, t+△t).

The congestion level process of this system is a process in continuous time. But for management science models, a fixed interval discrete time approximation to a continuous time process generally gives qualitative and quantitative results which are equivalent to the continous time processes. We formulate a two-stage tandem queueing service system as a Discrete-Time Markov Decision Process (DMDP) and the optimality criterion is the total discounted expected cost.

## RELATED RESEARCH

Up to decade ago, most researchers in queueing were concerned about descriptive analysis of queueing systems. They derived statistical characteristics, that is, stationary distributions, waiting time distribution, expected number of customers in the system, etc. But in all the branches of industry, managers are confronted wth problems of optimal design and control. A detailed survey of papers of design and control problems was done by Crabill, Gross, and Magazine (8). Some typical works are the following:

Crabill (7), Kakalik (18) and Sabeti (27) have considered an infinite waiting line single server system with Poisson arrival. The system has k possible exponential servicer ates. Each author used long-run expected average cost as the optimality criterion. Sabeti also considered discounted expected cost criterion,

Heyman (14) considered an infinite waiting line, a single channel queueing system in which the server can be turned-on or turned-off. Bell (1), Blackburn (2), Magazine (22) and Sobel (30) have considered related problems.

Deb and Serfozo (9) considered another on-off type of service facility. They assume that the server can make a decision to serve any number of customers in a batch, up to some batch size limit.

Magazine (23) and McGill (24) investigated a multi-channel system in which the control variable was the number of exponential servers to be used.

Hillier and Boling (15, 16), and Liaw (19) considered series production systems. They studied the characteristics of the system by a numerical method.

Nelson (27) and Taube-Netto (31) considered a two-stage tandem queueing facility which has infinite waiting capacity and a single server. The server can operate only one stage at a given time. They compared statistics for six different operating rules.

A control model with variable service rates for a closed queueing system was considered by Torberr (32). He tudied a sequential crew system as a cyclic queue.

In controlling the arrival process, Miller (25) considered a finite state model in which m classes of customers arrive at the system which has n servers and no waiting space. Each customer which is served results in a reward depending on his class. If a customer arrives when no server is free, he leaves without being served. Lippman and Ross (20) considered policies for accepting customers at a single service facility with no queue when the customers differ with respect to service. Low (21) considered a system in which customers must pay an entrance fee to join the queue. The entrance fee was controllable.

Evans (11) considered an assembly system which has an operator who assembles two parts to make one item of finished product. His model was developed two-dimensional process and the control variables were the arrival rates of both inputs. As far as we know, he was the first to examine a two-dimensional process in optimization of design and control of a queueing system. He also suggests various models which are multi-dimensional processes (12, 13).

In our judgement, we conclude that multi-dimensional process problems in optimization of design and control of queueing systems have received little attention compared to their appearance in real systems, perhaps because they are so complicated themselves. This was our main motivation to study a two-stage tandem queueing service system. The present work differs from the earlier works in that our model is a multi-dimensional process with the finite waiting capacity and series queueing system.

## MODEL FORMULATION

As shown in figure 1, we consider a two-stage tandem queueing service system which has two available service rates at each stage. Stage 1 has two available service rates $\alpha_1$ and $\alpha_2$ ($\alpha_2 > \alpha_1$), and Stage 2 has two available service rates, $\mu_1$ and $\mu_2$ ($\mu_2 > \mu_1$). Both Stage 1 and Stage 2 are controllable.

The manager controls both stages by selecting either of two service rates at each stage to minimize the total discounted expected cost.

## FORMULATION AS A DISCRETE-TIME MARKOV DECISION PROCESS (DMDP)

This system is formulated as a Discrete-Time Markov Decision process problem having:

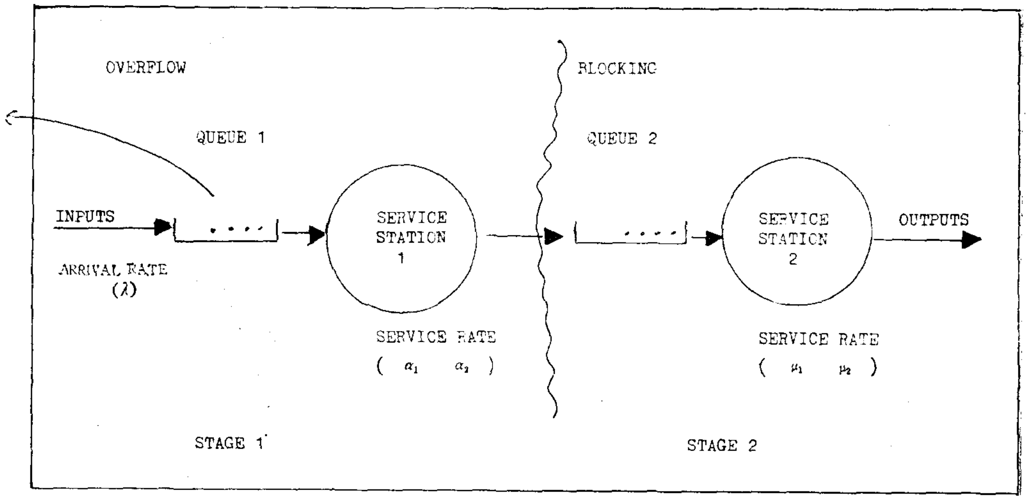


**FIGURE 1.** : Flow diagram of a tipical two-stage tandem queueing service system

1) **State Space S**

Let $S=\{(i,j)\mid i=0,1,\cdots\cdots, M-1, M, ;j=0,1,\cdots\cdots, N-1, N\}$

be the set of all possible states of the system where i and j represent the number of customers in Stage 1 and Stage 2 including the one in service, respectively.

2) **Action Space A**

The manager of this system has four possible actions which he can choose in each state $(i, j) \in S$.

$A=\{a\mid a=(1,1), (1,2), (2,1), (2,2)\}$

where $a=(a_1, a_2)$ means that the service rate $\alpha_{a_1}$ is employed at Stage 1 and the service rate $\mu_{a_2}$ is employed at Stage 2.

3) **Law of Motion**

Let $P_{ijkl}(a)$ be the transition probability that a state (i, j) at time t moves to a state (k, l) at time $(t+\triangle t)$ by choosing action $a\in A$.

If the system is in state $(i,j)\in S$ and action $a\in A$ is taken, then the transition probability $P_{ijkl}(a)$, $a=(a_1, a_2)$, is as follows:

$$P_{ijkl}(a) = \begin{cases} \lambda\varepsilon(i), & \text{for } (k,l)=(i+1,j), \\ \alpha_a\delta(i,j), & \text{for } (k,l)=(i-1,j+1), \\ \mu_a\phi(j), & \text{for } (k,l)=(i,j-1), \\ 1-\lambda\varepsilon(i)-\alpha_a\delta(i,j)-\mu_a\phi(j), & \\ & \text{for } (k,l)=(i,j), \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha_a=\alpha_{a_1}$, and $\mu_a=\mu_{a_2}$,

$$\varepsilon(i) = \begin{cases} 0 & \text{if } i\geq M \\ 1 & \text{otherwise} \end{cases}$$

$$\delta(i,j) = \begin{cases} 0 & \text{if } i=0 \text{ or } j\geq N \\ 1 & \text{otherwise} \end{cases}$$

$$\phi(j) = \begin{cases} 0 & \text{if } j=0 \\ 1 & \text{otherwise} \end{cases}$$

4) **Cost Structure**

The following cost structure is assumed:

Let

$r(\alpha_k)$ = operating cost per period at Stage 1 when service rete $\alpha_k$ $(k=1,2)$ is being used.

$r(\mu_k)$ = operating cost per period at Stage 2 when service rate $\mu_k$ $(k=1,2)$ is being used.

$-g$ = cost (−profit) of a unit completing service at Stage 2, that is, the system receives a gain, g, each time a unit completes service at Stage 2.

It is assumed that no switching costs when action is changed occur, and holding costs at Stage 1 and Stage 2 are not considered.

Let $w_{ijkl}$ a) be the cost incurred in time interval $(t, t+\triangle t)$, if the system moves from a state $(i,j)\in S$ at time t to a state $(k,l)\in S$ at time $(t+\triangle t)$ and action $a\in A$ is taken.

From the above assumptions, we have, for $a=(a_1, a_2)$,

$$W_{ijkl}(\mathrm{a})=\begin{cases} r(\alpha_a)+r(\mu_a), & \text{for } (k,l)=(i+1,j),\ (i+1,j),\ \text{and } (i,j), \\ r(\alpha_a)+r(\mu_a)-g, & \text{for } (k,l)=(i,j-1), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then, the expected one-period cost starting from a state (i, j) ∈S if action a∈A is chosen, is written as follows:

$$W_{ij}(a)=\sum_k \sum_l P_{ijkl}(a)\ W_{ijkl}(\mathrm{a})$$

$$=r(\alpha_a)+r(\mu_a)-g\mu_a\phi(j)$$

Now, we have the following dynamic programming problem:

For all (i, j) ∈S,

$$V_0(i,j)=0,$$

$$V_n(i,j)=\underset{a\in A}{\mathrm{Min}}\{r(\alpha_a)+r(\mu_a)-g\mu_a\phi(j)+\beta\{\lambda\varepsilon(i)\ V_{n-1}(i+1,j)$$

$$+\alpha_a\delta(i,j)\ V_{n-1}(i-1,j+1)+\mu_a\phi(j)\ V_{n-1}(i,j-1) \tag{1}$$

$$+(1-\lambda\varepsilon(i)-\alpha_a\delta(i,j)-\mu_a\phi(j))\ V_{n-1}(i,j))\}\}$$

Definition: Define

1) $x_n(i,j)=V_n(i,j)-V_n(i-1,j+1)$,

2) $y_n(i,j)=V_n(i,j)-V_n(i,j-1)$,

and

3) $z(i,j)=V_n(i,j)-V_n(i-1,j)$.

The recursive equation (1) can be written as follows:

For all (i, j) ∈S,

$$V_0(i,j)=0,$$

$$V_n(i,j)=\beta\lambda\varepsilon(i)\ V_{n-1}(i+1,j)+(1-\lambda\varepsilon(i))\ V_{n-1}(i,j)$$

$$+\underset{a\in A}{\mathrm{Min}}\{r(\alpha_a)+r(\mu_a)-g\mu_a\phi(j)-\beta\mu_a\phi(j)-\beta\mu_a\phi(j)\ y_{n-1}(i,j) \tag{2}$$

$$-\beta\alpha_a\delta(i,j)\ x_{n-1}(i,j)\}$$

**Properties of the n Period Optimal Policy**

Definition: Define f as a mapping function, which maps the finite state space S into the finite action space A. Thus f: S→A defines a decision rule and means that whenever the system is in state (i, j) ∈S, the action taken is f (i, j) ∈A. Define $f_n^*(i,j)$ as the n period optimal policy in state (i, j) ∈S.

We use the convention that in case of ties in the optimal values between actions, action (1, 1) is preferred to actions (1, 2), (2, 1) and (2, 2), action (1, 2) is preferred to actions (2, 1) and (2, 2), and action (2, 1) is preferved to action (2, 2).

First, we show that, in certain boundary states, certain control actions may be ignored because they can never be included in an optimal policy. This is done by proving the following lemmas.

Lemma 1: If Queue 1 and Queue 2 are empty, then it is optimal to use the slowest possible service rate at each stage, that is,

$$f_n^*(0,0)=(1,1) \text{ for all n.}$$

Proof: Using the facts that $r(\alpha_1)<r(\alpha_2)$ and $r(\mu_1)<r(\mu_2)$, the result is easily obtained.

Lemma 2: For all n, and all states $(0,j)\in S$, $(i,N)\in S$, $0<i\leq M$, $0<j\leq N$, action (1, 1) is superior to action (2, 1) and action (1, 2) is superior to action (2, 2), that is,

$$f_n^*(0,j)\neq(2,1),\ (2,2)$$

and

$$f_n^*(i,N)\neq(2,1),\ (2,2)$$

Proof: Using the optimal action $f_n^*$ and non-optimal action a, we have the following inequalities.

$$\begin{aligned}&r(f_n^*(0,j))-g\cdot\mu(f_n^*(0,j))-\beta\mu(f_n^*(0,j))y_{n-1}(0,j)\\&+\beta\lambda V_{n-1}(1,j)+\beta(1-\lambda)V_{n-1}(0,j)\\&\leq r(a)-g\mu(a)-\beta\mu(a)y_{n-}{}^{1}(o,j)\\&+\beta\lambda V_{n-1}(1,j)+\beta(1-\lambda)V_{n-1}(0,j) \qquad \text{for } 0<j\leq N,\end{aligned} \tag{3a}$$

and

$$\begin{aligned}&r(f_n^*(i,N))-g\mu(f_n^*(i,N))-\beta\mu(f_n^*(i,N)y_{n-1}(i,N)\\&+\beta\lambda V_{n-1}(i+1,N)+\beta(1-\lambda)V_{n-1}(i,N)\\&\leq r(a)-g\mu(a)-\beta\mu(a)y_{n-1}(i,N)+\beta\lambda V_{n-1}(i+1,N)\\&+\beta(1-\lambda)V_{n-1}(i,N) \qquad \text{for } 0<i<M\end{aligned} \tag{3b}$$

Note that by dropping all $\lambda$ terms in (3b), we have the inequality for the state (M, N).

Since $r(\alpha_1)<r(\alpha_2)$ and $r(\mu_1)<r(\mu_2)$, from inequalities (3a) and (3b), the results are obtained.

Lemma 3: For all n, and all states $(i,0)\in S$, $0<i\leq M$, action (1, 1) is superior to action (1, 2), and action (2, 1) is superior to action (2, 2), that is

$$f_n^*(i,0)\neq(1,2),\ (2,2)$$

Proof: The optimal action $f_n^*(i,0)$ satisfies the following inequality.

$$\begin{aligned}&r(f_n^*(i,0))+\beta\lambda V_{n-1}(i+1,0)+\beta(1-\lambda)V_{n-1}(i,0)\\&-\beta\alpha(f_n^*(i,0))\ x_{n-1}(i,0)\\&\leq r(a)+\beta\lambda V_{n-1}(i+1,0)+\beta(1-\lambda)V_{n-1}(i,0)\\&-\beta\alpha(a)x_{n-1}(i,0) \qquad \text{for } 0<i\leq M\end{aligned} \tag{4}$$

Using the inequality (4) and the fact that $r(\alpha_1)<r(\alpha_2)$, the result follows immediately. Note that for the state $(M,0)$, We obtain the inequality by dropping all $\lambda$ terms in (4).

In the following lemma, we present conditions which an optimal policy $f_n^*(.,.)$ must satisfy.

Lemma 4: Let

$$R_1=\frac{r(\alpha_2)-r(\alpha_1)}{\alpha_2-\alpha_1}$$

$$R_2=\frac{r(\mu_2)-r(\mu_1)}{\mu_2-\mu_1}$$

Then, for all n, and each $(i,j)\in S$,

1) $f_n^*(i,j)=(1,1)$, if and only if

$$\beta y_{n-1}(i,j)\phi(j)\leq R_2-g,$$

$$\beta x_{n-1}(i,j)\delta(i,j) \leq R_1, \tag{5a}$$

and

$$\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)\delta(i,j)-\beta(\mu_2-\mu_1)y_{n-1}(i,j)\phi(j)$$
$$\leq r(\alpha_2)-r(\alpha_1)+r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1),$$

2) $f_n^*(i,j)=(1,2)$, if and only if

$$\beta y_{n-1}(i,j)\phi(j)>R_2-g$$
$$\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)\delta(i,j)-\beta(\mu_2-\mu_1)y_{n-1}(i,j)\phi(j)$$
$$\leq r(\alpha_2)-r(\alpha_1)-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1), \tag{5b}$$

and

$$\beta x_{n-1}(i,j)\delta(i,j) \leq R_1,$$

3) $f_n^*(i,j)=(2,1)$, if and only if

$$\beta x_{n-1}(i,\ )\delta(i,j)>R_1,$$
$$\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)\delta(i,j)-\beta(\mu_2-\mu_1)y_{n-1}(i,j)\phi(j) \tag{5c}$$
$$>r(\alpha_2)-r(\alpha_1)-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1),$$

and

$$\beta y_{n-1}(i,j)\phi(j) \leq R_2-g,$$

4) $f_n^*(i,j)=(2,2)$, if and only if

$$\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)\delta(i,j)+(\mu_2-\mu_1)y_{n-1}(i,j)\phi(j) \tag{5d}$$
$$>r(\alpha_2)-r(\alpha_1)+r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1),$$
$$\beta x_{n-1}(i,j)\delta(i,j)>R_1,$$

and

$$\beta y_{n-1}(i,j)\phi(j)>R_2-g$$

proof: Since the proof of each boundary state is quite similar to the general state, we shall show here only the proof for the general state (i, j), 0<i<M, 0<j<N.

The optimal action $f_n^*(i,j)$ satisfies the following inequlaity.

$$r\left(\alpha_{f_n^*(i,j)}\right)+r\left(\mu_{f_n^*(i,j)}\right)-g\mu_{f_n^*(i,j)}-\beta\alpha_{f_n^*(i,j)}x_{n-1}(i,j)$$
$$-\beta\mu_{f_n^*(i,j)}y_{n-1}(i,j)+\beta\lambda V_{n-1}(i+1)+\beta(1-\lambda)V_{n-1}(i,j) \tag{5e}$$
$$\leq r(\alpha_a)+r(\mu_a)-g\mu_a-\beta\alpha_a x_{n-1}(i,j)-\beta\mu_a y_{n-1}(i,j)$$
$$+\beta\lambda V_{n-1}(i+1,j)+\beta(1-\lambda)V_{n-1}(i,j)$$

Using the inequality (5e), we see that $f_n^*(i,j)=(1,1)$ if and only if

$$r(\alpha_1)+r(\mu_1)-g\mu_1-\beta\alpha_1 x_{n-1}(i,j)-\beta\mu_1 y_{n-1}(i,j)$$
$$\leq r(\alpha_1)+r(\mu_2)-g\mu_2-\beta\alpha_1 x_n\text{-}1(i,j)-\beta\mu y_{n-1}(i,j), \quad \text{for } a=(1,2),$$
$$\leq r(\alpha_2)+r(\mu_1)-g\mu_{1\text{-}}\beta\alpha_2 x_{n-1}(i,j)-\beta\mu,\ y_{n-1}(i,j), \quad \text{for } a=(2,1),$$

and

$$\leq r(\alpha_2)+r(\mu_2)-g\mu_2-\beta\alpha_2 x_{n-1}(i,j)-\beta\mu_2 y_{n-1}(i,j), \quad \text{for } a=(2,2)$$

Simplifying these inequalities, we obtain (5a).

Thus, $f_n^*(i,j)=(1,1)$, then (5a) hold, and, if (5a) is true, then $f_n^*(i,j)=(1,1)$.

The cases for $f_n^*(i,j)=(1,2)$, $f_n^*(i,j)=(2,1)$, and $f_n^*(i,j)=(2,2)$, are p roven similarly. Thus proofs are omitted.

Next, we present the main characterisics of th n period optimal policy. It is our attempt to show that the n period optimal policy $f_n^*(.,.)$ is a four connected region policy as shown in figure 2. It has the property that the state space S is broken into four connected subsets.

Definition: Let I denote the set of non-negative integers. A function $V(i, j)$ is nondecreasing (nonincreasing) in $i \in I$ if for all $i_1, i_2 \in I$,

$$V(i_1, j) \geq V(i_2, j) \quad (V(i_1, j) \leq V(i_2, j).$$

This definition is also applied in $j \in I$.

Definition: Oefine $H_n\{(i, j), a_1, a_2\}$ as the difference of return value when action $a_1$and action $a_2$ are taken in starting state $(i, j)$, n period remaining, that is, $\{H_n(i, j), a_1, a_2\}$ is defined as follows :

$$H_n\{(i, j), a_1, a_2\} = V_n^{a_1}(i, j) - v_n^{a_2}(i, j).$$

Note that $H_n\{(i, j), f_n^*(i, j), a\} \leq 0$ where a is non-optimal action.

Definition: Define the following properties of the optimal value function.

P1: $y_n(i, j) = V_n(i, j) - V_n(i, j-1)$ is nondecreasing function of $j$,

P2: $y_n(i, j) = V_n(i, j) - V_n(i, j\text{-}1)$ is nondecreasing function of $j$,

P3, $x_n(i, j) = V_n(i, j) - V_n(i\text{-}1, j+1)$ is nonincreasing function of $j$,

P4: $x_n(i, j) = V_n(i, j) - V_n(i\text{-}1, j+1)$ is nondecreasing function of $i$,

P5: $x_n(i, j) = V_n(i, j) - V_n(i-1, j+1) \geq 0$,

P6: $y_n(i, j) = V(i, j) - V_n(i, j-1) > -g$,

and

P7: $z_n(i, j) = V(i, j) - V_n(i-1, j) \leq 0$.

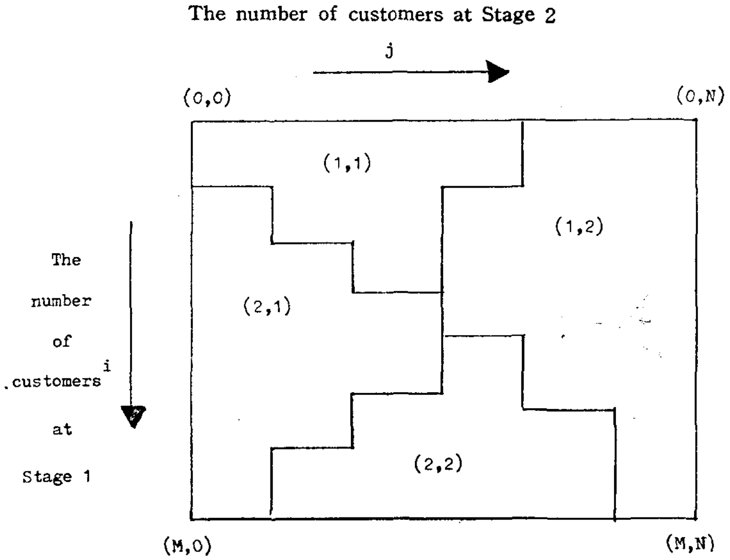


**Figure** 2: Form of the n Period Cptimal Policy of Both Stages Control Model with Two Available Serice Rates at Each Stage

In Lemma 1–4, we discussed part of the charcteristics of the n period optimal policy. The ollowing five theorems describe the main charaterististics of the n period optimal policy.

Theorem 1: Assume that $V_{n-1}(i, j)$, $(i, j) \in S$, satisfies all of properties P1–P7.

If there is a state $(i,j)$ such that $f_n^*(i,j)=(1,2)$, then, $f_n^*(i,j')=(1,2)$ for $j'>j$.

Proof: Assume that there is a state $(i,j)$ such that $f_n^*(i,j)=(1,2)$. Since $f_n^*(i,j)=(1,2)$,

$$H_n\{(i,j),\ (1,2),\ (1,1)\}$$
$$=r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1)-\beta(\mu_2-\mu_1)y_{n-1}(i,j)<0, \tag{6a}$$
$$H_n\{(i,j),\ (1,2),\ (2,1)\}$$
$$=r(\mu_2)-r(\mu_1)-r(\alpha_2)+r(\alpha_1)-g(\mu_2-\mu_1)$$
$$+\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)-\beta(\mu_2-\mu_1)y_{n-1}(i,j)\leq 0, \tag{6b}$$

and

$$H_n\{(i,j),\ (1,2),\ (2,2)\}$$
$$=r(\alpha_1)-r(\alpha_2)+\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)\leq 0. \tag{6c}$$

we claim that, for $j'=j+1,\ j+2,\ldots\ldots,N-1,\ N$,

$$H_n\{(i,j'),\ (1,2),\ (1,1)\}<0,$$
$$H_n\{(i,j'),\ (1,2),\ (2,1)\}\leq 0,$$

and

$$H_n\{(i,j'),\ (1,2),\ (2,2)\}\leq 0.$$

Starting with $j'=j+1$, for the state $(i,j+1)\in S$,

we have

$$H_n\{(i,j+1),\ (1,2),\ (1,1)\}$$
$$=r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1)-\beta(\mu_2-\mu_1)y_{n-1}(i,j+1), \tag{7a}$$
$$H_n\{(i,j+1),\ (1,2),\ (2,1)\}$$
$$=r(\mu_2)-r(\mu_1)-r(\alpha_2)+r(\alpha_1)-g(\mu_2-\mu_1)$$
$$+\beta(\alpha_2-\alpha_1)x_{n-1}(i,j+1),-\beta(\mu_2-\mu_1)y_{n-1}(i,j+1), \tag{7b}$$

and

$$H_n\{(i,j+1),\ (1,2),\ (2,2)\}$$
$$=r(\alpha_1)-r(\alpha_2)+\beta(\alpha_2-\alpha_1)x_{n-1}(i,j+1) \tag{7c}$$

Subtracting (6) from (7), respectively,

we obtain,

$$H_n\{(i,j+1),\ (1,2)\ (1,1)\}-H_n\{(i,j),\ (1,2),\ (1,1)\}$$
$$=-\beta(\mu_2-\mu_1)y_{n-1}(i,j)-y_{n-1}(i,j)<0, \tag{8a}$$
$$H_n\{(i,j+1),\ (1,2),\ (2,1)\}-H_n\{(i,j),\ (1,2),\ (2,1)\}$$
$$=\beta(\alpha_2-\alpha_1)\{x_{n-1}(i,j+1)-x_{n-1}(i,j)\}$$
$$-\beta(\mu_2-\mu_1)\{y_{n-1}(i,j+1)-y_{n-1}(i,j)\}\leq 0, \tag{8b}$$

and

$$H_n\{(i,j+1),\ (1,2),\ (2,2)\}-H_n\{(i,j),\ (1,2),\ (2,2)\}$$
$$=\beta(\alpha_2-\alpha_1)\{x_{n-1}(i,j+1)-x_{n-1}(i,j)\}\leq 0. \tag{8c}$$

Since from the assumptions, for all $(i,j)\in S$,

$$x_{n-1}(i,j)-x_{n-1}(i,j-1)\leq 0,$$

and

$$y_{n-1}(i,j)-y_{n-1}(i,j-1)\geq 0.$$

Thus, from (8a−8c), we obtain the desired result,

$$H_n\{(i,j+1),\ (1,2),\ (1,1)\}<0,$$

$$H_n\{(i,j+1),\ (1,2),\ (2,1)\}\leq 0,$$

and

$$H_n\{(i,j+1),\ (1,2),\ (2,2)\}\leq 0,$$

This implies that $f_n^*(i,j+1)=(1,2)$.

Continuing in this manner, we can show that $f_n^*(i,j')=(1,2)$, for $j'=j+1, j+2, \ldots\ldots, N-1, N$

Theorem: 2 Assume that $V_{n-1}(i,j)$, $(i,j)\in S$, satisfies all of properties P1—P7.

If there is a state $(i,j)$ such that $f_n^*\ (i,j)=(1,1)$, then, $f_n^*\ (i',j)=(1,1)$ for $i'<i$.

Proof: Assume that there is a state $(i,j)$ such that $f_n^*(i,j)=(1,1)$. Since $f_n^*(i,j)=(1,1)$, we have,

$$H_n\{(i,j),\ (1,1),\ (1,2)\}$$
$$=-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1)+\beta(\mu_2-\mu_1)y_{n-1}(i,j)\leq 0, \tag{9a}$$
$$H_n\{(i,j),\ (1,1),\ (2,1)\}$$
$$=-r(\alpha_2)+r(\alpha_1)r(\alpha_1-\alpha_1)x_{n-1}(i,j)\leq 0, \tag{9b}$$
$$H_n\{(i,j),\ (1,1),\ (2,2)\}$$
$$=-r(\alpha_2)+r(\alpha_1)-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1)$$
$$+\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)+\beta(\mu_2-\mu_1)y_{n-1}(i,j)\leq 0. \tag{9c}$$

We claim that, for $i'=i-1,\ i-2, \ldots\ldots 1, 0$,

$$H_n\{(i',j),\ (1,1),\ (1,2)\}\leq 0,$$
$$H_n\{(i',j),\ (1,1),\ (2,1))\leq 0,$$

and

$$H_n\{(i',j),\ (1,1),\ (2,2)\}\leq 0,$$

Starting with $i'=i-1$, for a state $(i-1,j)\in S$,

we have,

$$H_n\{(i-1,j),\ (1,1),\ (1,2)\}$$
$$=-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1)+\beta(\mu_2-\mu_1)y_{n-1}(i-1,j), \tag{10a}$$
$$H_n\{(i-1,j),\ (1,1),\ (2,1)\}$$
$$=-r(\alpha_2)+r(\alpha_1)+\beta(\alpha_2-\alpha_1)x_{n-1}(i-1,j), \tag{10b}$$

and

$$H_n\{(i-1,j)\ (1,1),\ (2,2)\}$$
$$=-(\alpha_2)+r(\alpha_1)-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1)$$
$$+\beta(\alpha_2-\alpha_1)x_{n-1}(i-1,j)+\beta(\mu_2-\mu_1)y_{n-1}(i-1,j). \tag{10c}$$

Subtracting (9) from (10) and using the assumptions

$$y_{n-1}(i,j)-y_{n-1}(i-1,j)\geq 0$$

and

$$x_{n-1}(i,j)-x_{n-1}(i-1,j)\geq 0 \qquad \text{for all } (i,j)\in S,$$

we obtain,

$$H_n\{(i-1,j),\ (1,1),\ (1,2)\}-H_n\{(i,j),\ (1,1),\ (1,2)\}$$
$$=\beta(u_2-u_1)\ \{y_{n-1}(i-1,j)-y^{u}_{-1}(i,j)\}\leq 0, \tag{11a}$$
$$H_n\{(i-1,j),\ (1,1),\ (2,1)\}-H_n\{i,j),\ (1,1),\ (2,1)\}$$

$$=\beta(\alpha_2-\alpha_1)\{x_{n-1}(i-1,j)-x_{n-1}(i,j)\}\leq 0, \quad (11b)$$

$$H_n\{(i-1,j),(1,1),(2,2)\}-H_n\{i,j),(1,1),(2,2)\}$$
$$=\beta(\alpha_2-\alpha_1)\{x_{n-1}(i-1,j)-x_{n-1}(i,j)\}$$
$$+\beta(\mu_2-\mu_1)\{y_{n-1}(i-1,j)-y_{n-1}(i,j)\}\leq 0, \quad (11c)$$

From (11a−11c), we have

$$H_n\{(i-1,j),\ (1,1),\ (1,2)\}\leq 0,$$
$$H_n\{(i-1,j),\ (1,1),\ (2,1)\}\leq 0,$$

and

$$H_n\{(i-1,j),\ (1,1),\ (2,2)\}\leq 0.$$

Thus we obtain the desired result, $f_n^*(i-1,j)=(1,1)$.
Using the same method, we can show that $f_n^*(i',j)=(1,1)$, for $i'=i-2,i-3,\ \ldots\ldots.1,0.$
Thus the proof is completed.

Theorem 3: Assume that $V_{n-1}(i,j)$, $(i,j)\in S$, satisfies all of properties P1—P7.
If there is a state $(i,j)$ such that $f_n^*(i,j)=(2,1)$, then, $f_n^*(i,j')=(2,1)$ for $j'<j$.
Proof: Assume that there is a state $(i,j)$ such that $f_n^*(i,j)=(2,1)$. Since $f_n^*(i,j)=(2,1)$, we have,

$$H_n\{(i,j),\ (2,1),\ (1,1)\}$$
$$=r(\alpha_2)-r(\alpha_1)-\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)<0, \quad (12a)$$
$$H_n\{(i,j),\ (2,1),\ (1,2)\}$$
$$=-r(\mu_2)+r(\mu_1)+r(\alpha_2)-r(\alpha_1)+g(\mu_2-\mu_1)$$
$$-\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)+\beta(\mu_2-\mu_1)y_{n-1}(i,j)<0, \quad (12b)$$

and

$$H_n\{(i,j),\ (2,1),\ (2,2)\}$$
$$=-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1)+\beta(\mu_2-\mu_1)y_{n-1}(i,j)\leq 0, \quad (12c)$$

we want to show that, for $j'=j-1,j-2,\ \ldots\ldots\ldots\ 1,0,$

$$H_n\{(i,j'),\ (2,1),\ (1,1)\}<0,$$
$$H_n\{(i,j'),\ (2,1),\ (1,2)\}<0,$$

and

$$H_n\{(i,j'),\ (2,1),\ (2,2)\}\leq 0.$$

Starting with $j'=j-1$, for a state $(i,j-1)\in S$,
we have,

$$H_n\{(i,j-1),\ (2,1),\ (1,1)\}$$
$$=r(\alpha_2)-r(\alpha_1)-\beta(\alpha_2-\alpha_1)x_{n-1}(i,j-1), \quad (13a)$$
$$H_n\{(i,j-1),\ (2,1),\ (1,2)\}$$
$$=-r(\mu_2)+r(\mu_1)+r(\alpha_2)-r(\alpha_1)+g(\mu_2-\mu_1)$$
$$-\beta(\alpha_2-\alpha_1)x_{n-1}(i,j-1)+\beta(\mu_2-\mu_1)x_{n-1}(i,j-1), \quad (13b)$$

and

$$H_n\{i,j-1),\ (2,1),\ (2,2)\}$$
$$=-r(\mu_2)+r(\mu_1)+g(\mu_2-\mu_1)+\beta(\mu_2-\mu_1)y_{n-1}(i,j-1), \quad (13b)$$

Subtracting (12) from (13), and using the assumtions

$$x_{n-1}(i,j)-x_{n-1}(i,j-1)\leq 0,$$

and

$$y_{n-1}(i,j)-y_{n-1}(i,j-1)\geq 0 \qquad \text{for all } (i,j)\in S,$$

We obtain,

$$H_n\{(i,j-1),\ (2,1),\ (1,1)\}-H_n\{(i,j),\ (2,1),\ (1,1)\}$$
$$=-\beta(\alpha_2-\alpha_1)\{x_{n-1}(i,j-1)-x_{n-1}(i,j)\}<0, \tag{14a}$$
$$H_n\{(i,j-1),\ (2,1),\ (1,2)\}-H_n\{(i,j),\ (2,1),\ (1,2)\}$$
$$=-\beta(\alpha_2-\alpha_1)\{x_{n-1}(i,j-1)-x_{n-1}(i,j)\}$$
$$+\beta(\mu_2-\mu_1)\{y_{n-1}(i,j-1)-y_{n-1}(i,j)\}<0, \tag{14b}$$

and

$$H_n\{(i,j-1),\ (2,1),\ (2,2)\}-H_n\{(i,j),\ (2,1),\ (2,2)\}$$
$$=\beta(\mu_2-\mu_1)\{y_{n-1}(,j,-1)-y_{n-1}(i,j)\}\leq 0. \tag{14c}$$

From (14), we have,

$$H_n\{i,j-1),\ (2,1),\ (1,1)\}<0,$$
$$H_n\{i,j-1),\ (2,1),\ (1,2)\{<0,$$

and

$$H_n\{i,j-1),\ (2,1),\ (2,2)\}\leq 0.$$

Thus we obtain the desired result, $f_n^*(i,j-1)=(2,1)$

Continuing in this manner, we can show that $f_n^*(i,j')=(2,1)$, for $j'=j-2,\ j-3,\ldots.1,0.$

Theorem 4: Assume that $V_{n-1}(i,j)$, $(i,j)\in S$, satisfies all of properties P1～P7.

If there is a state $(i,j)$ such that $f_n^*(i,j)=(2,2)$, then, $f_n^*(i',j)=(2,2)$ for $i'>i$.

Proof: Assume that there is a state $(i,j)$ such that $f_n^*(i,j)=(2,2)$. We want to show that, for $i'=i+1,\ i+2,\ldots\ldots.,\ M-1,\ M$,

$$H_n\{(i',j),\ (2,2),\ (1,1)\}<0,$$
$$H_n\{(i',j),\ (2,2),\ (1,2)\}<0,$$

and

$$H_n\{(i',j),\ (2,2),\ (2,1)\}<0.$$

Since $f_n^*(i,j)=(2,2)$, we have,

$$H_n\{(i,j),\ (2,2),\ (1,1)\}$$
$$=r(\alpha_2)-r(\alpha_1)+r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1)$$
$$-\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)-\beta(\mu_2-\mu_1)y_{n-1}(i,j)<0, \tag{15a}$$
$$H_n\{(i,j),\ (2,2),\ (1,2)\}$$
$$=r(\alpha_2)-r(\alpha_1)-\beta(\alpha_2-\alpha_1)x_{n-1}(i,j)<0, \tag{15b}$$

and

$$H_n\{(i,j),\ (2,2),\ (2,1)\}$$
$$=r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1)y_{n-1}(i,j)<0. \tag{15c}$$

Starting with $i'=i+1$, for a state $(i+1,j)\in S$,

we have,

$$H_n\{(i+1,j),\ (2,2),\ (1,1)\}$$
$$=r(\alpha_2)-r(\alpha_1)+r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1)$$
$$-\beta(\alpha_2-\alpha_1)x_{n-1}(i+1,j)-\beta(\mu_2-\mu_1)y_{n-1}(i+1,j), \tag{16a}$$
$$H_n\{(i+1,j),\ (2,2),\ (1,2)\}$$
$$=r(\alpha_2)-r(\alpha_1)-\beta(\alpha_2-\alpha_1)x_{n-1}(i+1,j), \tag{16b}$$

and

$$H_n\{(i+1,j),\ (2,2),\ (2,1)\}$$
$$=r(\mu_2)-r(\mu_1)-g(\mu_2-\mu_1)-\beta(\mu_2-\mu_1)y_{n-1}(i+1,j). \tag{16c}$$

Subtracting (15) from (16), and using the assumptions

$$x_{n-1}(i,j)-x_{n-1}(i-1,j)\geq 0,$$

and

$$y_{n-1}(i,j)-y_{n-1}(i-1,j)\geq 0 \qquad \text{for all } (i,j)\in S,$$

we obtain

$$H_n\{(i+1,j),\ (2,2),\ (1,1)\}-H_n\{(i,j),\ (2,2),\ (1,1)\}$$
$$=-\beta(\alpha_2-\alpha_1)\{x_{n-1}(i+1,j)-x_{n-1}(i,j)\}$$
$$-\beta(\mu_2-\mu_1)\{y_{n-1}(i+1,j)-y_{n-1}(i,j)\}<0, \tag{17a}$$
$$H_n\{(i+1,j),\ (2,2),\ (1,2)\}-H_n\{(i,j),\ (2,2),\ (1,2)\}$$
$$=-\beta(\alpha_2-\alpha_1)\{x_{n-1}(i+1,j)-x_{n-1}(i,j)\}<0. \tag{17b}$$

and

$$H_n\{(i+1,j),\ (2,2),\ (2,1)\}-H\{(i,j),\ (2,2),\ (2,1)\}$$
$$=-\beta(\mu_2-\mu_1)\{y_{n-1}(i+1,j)-y_{n-1}(i,j)\}<0. \tag{17c}$$

From (17), we have,

$$H_n\{(i+1,j),\ (2,2),\ (1,1)\}<0,$$
$$H_n\{(i+1,j),\ (2,2),\ (1,2)\}<0,$$
$$H_n\{(i+1,j),\ (2,2),\ (2,1)\}<0.$$

Thus, we obtain the desired result. $f_n^*(i+1,j)=(2,2)$.
Continuing the same procedure, we can show that $f_n^*(i',j)=(2,2)$, for $i'=i+2,\ i+3,\ldots\ldots\ldots\ldots,M-1,\ M$.

Theorem 5: Assume that $V_n(i,j)$, $(i,j)\in S$, satisfies all of properties P1−P7.
If $f_n^*(i,j)=(2,1)$, then $f_n^*(i-1,j)\neq(1,2)$

Proof: The proof is by contradiction.
Assume that there exist states $(i,j)$ and $(i-1,j)$ such that $f_n^*(i,j)=(2,1)$, and $f_n^*(i-1,j)=(1,2)$.
Then, from Lemma 4, we have

$$y_{n-1}(i,j)\leq R_2-g, \quad (\text{since } f_n^*(i,j)=(2,1)),$$

and

$$y_{n-1}(i-1,j)>R_2-g, \quad (\text{since } f_n^*(i,j)=(1,2)),$$

Combining these two inequalities, we obtain

$$y_{n-1}(i,j)-y_{n-1}(i-1,j)\leq 0.$$

But this is contradiction to that $V_{n-1}(i, j)$, $(i, j) \in S$, satisfies the property P2, i.e., $y_{n-1}(i, j) - y_{n-1}(i-1, j) \geq 0$.

Thus,

$$f_n^*(i-1, j) \neq (1, 2)$$

The properties P1—P7 of the optimal value function $V_n(i, j,)$, $(i, j) \in S$ can be proved by inductive method. But because the system has nine boundary states and proofs are required rather lengthy analyses, so proofs will be omitted here.

## COMPUTATIONAL EXPERIENCE

Example 1:

optimal policy:

| States | j 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | (1, 1) | (1, 1) | (1, 1) | (1, 2) | (1, 2) | (1, 2) |
| 1 | (2, 1) | (1, 1) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |
| 2 | (2, 1) | (2, 1) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |
| 3 | (2, 1) | (2, 1) | (2, 2) | (1, 2) | (1, 2) | (1, 2) |
| 4 | (2, 1) | (2, 1) | (2, 2) | (2, 2) | (1, 2) | (1, 2) |
| 5 | (2, 1) | (2, 1) | (2, 2) | (2, 2) | (1, 2) | (1, 2) |
| *i* 6 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (1, 2) | (1, 2) |
| 7 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (1, 2) | (1, 2) |
| 8 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (2, 2) | (1, 2) |
| 9 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (2, 2) | (1, 2) |

test data:

$\lambda=0.10$

$\alpha_1=0.15$ $\alpha_2=0.30$

$\mu_1=0.25$ $\mu_2=0.30$

$r(\alpha_1)=3.0$ $r(\alpha_2)=5.0$

$r(\mu_1)=4.0$ $r(\mu_2)=5,0$

$g=100.0$ $n=90$

$\beta=0.98$

$M=9$ $N=5$

Example 2:

optimal policy:

| States | j 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | (1, 1) | (1, 2) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |
| 1 | (1, 1) | (1, 2) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |
| 2 | (1, 1) | (1, 2) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |
| 3 | (2, 1) | (1, 2) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | (2, 1) | (2, 2) | (1, 2) | (1, 2) | (1, 2) | (1, 2) |
| 5 | (2, 1) | (2, 2) | (2, 2) | (1, 2) | (1, 2) | (1, 2) |
| *i* 6 | (2, 1) | (2, 2) | (2, 2) | (1, 2) | (1, 2) | (1, 2) |
| 7 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (1, 2) | (1, 2) |
| 8 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (2, 2) | (1, 2) |
| 9 | (2, 1) | (2, 2) | (2, 2) | (2, 2) | (2, 2) | (1, 2) |

test data:

$\lambda=0.10$

$\alpha_1=0.25$ $\alpha_2=0.30$

$\mu_1=0.25$ $\mu_2=0.40$

$r(\alpha_1)=4.0$ $r(\alpha_2)=5.0$

$r(\mu_1)=4.0$ $r(\mu_2)=6.0$

$g=100.0$ $n=90$

$\beta=0.98$

$M=9$ $N=5$

## CONCLUSIONS

In the above, analytical and numerical results have been presented for a two-stage tandem queueing service system control model. The optimal dynamic operating policy was characterized when the optimality criterion was the total discounted expected cost.

We found out that, from our experience in numerical work, under specified conditions for the parameters, the form of the optimal policy has a much simpler connected region than the general forms of figure 2. More specific and simple forms of the optimal policy of two-stage tandem queueing service system may be obtained under restriction of certain condition in the parameters.

Suppose that each control model of the previous chapters has k available sevice rates. The optimal dynamic operating policy for each model with the added assumption can be characterized. As in Crabill's result (7), certain control actions among k possible actions may be ignored because they can never be included in an optimal policy; that is, certain non-optimal control actions may be eliminated from consideration.

Finally, consider a a series network production system with n service stations in series and k service crews. The question is "how should the manager dynamically allocate his labor force among n service stations?" The optimal dynamic allocation of service crews is a meaningful optimization problem. Control models for two-stage tandem queueing service system should be extended to the general n-stage tandem queueing service system. Of course, the degree of difficulty to analyze the system increases as the number of stages increase.

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