NOTE ON THE PRIME RADICAL IN NONASSOCIATIVE RINGS

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1. The prime radical

Several definitions of prime ideals in a nonassociative ring have been introduced during the past decade. An axiomatic definition, based on a *-operation, is given in [3] and extends most of the known results for the prime radical [4]. A *-operation in a nonassociative ring $R$ is a mapping of $I(R) \times I(R)$, where $I(R)$ is the lattice of ideals in $R$, into the lattice of additive subgroups of $R$ such that, for $A, B, C, D \in I(R)$, 1) if $A \subseteq C$ and $B \subseteq D$, then $A*B \subseteq C*D$, 2) $(0)*A=B*(0)=(0)$, and 3) if $R$ is a homomorphic image of $R$, then $A*B=A*B$.

An ideal $P$ of $R$ is called *-prime if $A*B \subseteq P$ for $A, B \in I(R)$ implies that $A \subseteq P$ or $B \subseteq P$. A nonempty subset $M$ of $R$ is called a *-system if, for $A, B \in I(R)$, $M \cap A \neq \phi$ and $M \cap B \neq \phi$ imply $A*B \cap M \neq \phi$. The *-prime radical $P^*(R)$ of $R$ is defined to be the set of elements $x \in R$ such that any *-system containing $x$ also contains 0 and shown to be the intersection of all *-prime ideals in $R$. If $R$ is an $s$-ring for a positive integer $s \geq 2$, then there exists a *-operation in $R$ such that $A*A=N$ for all $A \in I(R)$ and $P^*(R)$ coincides with the prime radical $P(R)$ of $R$ defined by Zwier [8] [4].

Let $A*B=AB^2+(AB)B+B(AB)+(BA)B+B(BA)+B^2A$ for $A, B \in I(R)$. Then, in a weakly W-admissible ring $R$, $A*B$ is also an ideal of $R$ [7]. The proof of Smith [6, Lemma 2.3] can be applied to show that $A*B$ for $A, B \in I(R)$ is an ideal in a generalized alternative ring II. Thus these rings are 3-rings which generalize Lie, alternative, Jordan, standard, and generalized standard rings. If we let $A \circ B=AB+BA$ for $A, B \in I(R)$, it is shown that $A \circ B$ is an ideal in Lie, alternative and $(-1, 1)$ rings [1] (in the alternative case, $AB$ is an ideal). Hence these rings are 2-rings. In a 2-ring we have following

**Proposition 1.** Let $A*B=AB^2+B^2A+(AB)B+B(AB)+(BA)B+B(BA)$ and $A \circ B=AB+BA$ for $A, B \in I(R)$. In a 2-ring $R$, an ideal $P$ of $R$ is prime if and only if $P$ is *-prime if and only if $P$ is $\circ$-prime.

**Proof.** Let $P$ be prime and let $A*B \subseteq P$ for $A, B \in I(R)$. Then $AB^2 \subseteq A*B \subseteq P$. Since $B^2$ is an ideal of $R$ and $P$ is prime, $A \subseteq P$ or $B \subseteq P$ and so $P$
is \(*\)-prime. Suppose that \(P\) is \(*\)-prime and \(AB \subseteq P\). Let \(C = A \cap B\). Then \(C \cdot C = C^2 \subseteq A^2 B + A B^2 \subseteq AB \subseteq P\) and so \(C \subseteq P\). Thus \(A \cdot B \subseteq A \cap B = C \subseteq P\), and since \(P\) is \(*\)-prime, \(A \subseteq P\) or \(B \subseteq P\) and \(P\) is prime. If \(P\) is prime, it is clearly \(\ast\)-prime. Suppose that \(P\) is \(\ast\)-prime and \(AB \subseteq P\). By a similar argument, if we let \(C = A \cap B\), then \(C \cdot C = C^2 \subseteq AB \subseteq P\) and so \(C \subseteq P\). It follows from this that \(BA \subseteq A \cap B = C \subseteq P\) and \(A \oplus B \subseteq P\). Thus \(P\) is prime.

Proposition 1 has been proved in a ring \(R\) in which \(AB\) is an ideal for \(A, B \in I(R)\) [4, Lemma 3.2].

In this note we give an analogous characterization of the prime radical in an \(s\)-ring for the \(*\)-prime radical of any rings [5]. We make use of this to show that, in an \(s\)-ring \(R\), every nonzero ideal of \(R\) which is contained in the prime radical of \(R\) contains a nonzero ideal \(K\) of \(R\) such that \(K^s = 0\), and that the prime radical of \(R\) is essentially nilpotent. This extends the result of Fisher [2] for associative rings to any \(s\)-ring.

2. Characterization of the prime radical

Following Rich [5], we make

**Definition 1.** Let \(R\) be any ring equipped with a \(*\)-operation. A sequence \(\{a_0, a_1, \ldots, a_n, \ldots\}\) in \(R\) is called a \(P^*\)-sequence if \(a_n \in (a_{n-1})^* (a_{n-1})^*_{n} \) for \(n = 1, 2, \ldots\). An element \(a\) of \(R\) is called strongly \(*\)-nilpotent if every \(P^*\)-sequence beginning with \(a\) is ultimately 0.

If \(R\) is an \(s\)-ring in which \(A^* A = A^r\) for \(A \in I(R)\), then the \(P^*\)-sequences are the \(P\)-sequences in [5].

**Theorem 2.** The \(*\)-prime radical \(P^*(R)\) of any ring \(R\) is the set of all strongly \(*\)-nilpotent elements in \(R\).

**Proof.** Let \(a\) be an element in \(R\) but not in \(P^*(R)\). Then there exists a \(\ast\)-prime ideal \(P\) of \(R\) which does not contain \(a\). The complement \(c(P)\) of \(P\) is a \(\ast\)-system in \(R\). Let \(a_0 = a\). Since \((a_0) \cap c(P) \neq \emptyset\), there exists a nonzero element \(a_1\) in \((a_0)^* (a_0) \cap c(P)\), and we inductively find a sequence \(S = \{a_0, a_1, \ldots, a_n, \ldots\}\) in \(R\) such that \(a_{n+1} \in (a_n)^* (a_n) \cap c(P)\). Thus \(S\) is a \(P^*\)-sequence beginning with \(a\) which does not end in zero, so that \(a\) is not strongly \(*\)-nilpotent in \(R\).

Conversely, suppose that \(a \in P^*(R)\) and that \(S = \{a_0, a_1, \ldots, a_n, \ldots\}\), where \(a_0 = a\), is a \(P^*\)-sequence beginning with \(a\). Let \(A, B\) be ideals of \(R\) such that \(A \cap S \neq \emptyset\) and \(B \cap S \neq \emptyset\). There exist elements \(a_{i_1} \in A \cap S\), \(a_{i_2} \in B \cap S\). Let \(j = \max \{i_1, i_2\}\). Then \(a_{j+1} \in (a_j)^* (a_j) \subseteq (a_{i_1})^* (a_{i_2}) \subseteq A^* B\). Thus \(a_j \in A^* B \cap S \neq \emptyset\), and this shows that \(S\) is a \(\ast\)-system in \(R\). Since \(a \in S \cap P^*(R)\), \(S\) must
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contain 0. Hence $a_j=0$ for some $j$ and $a$ is strongly $*$-nilpotent.

**Corollary 3.** Let $J$ be a nonzero ideal of $R$ which is contained in $P^*(R)$. For every $*$-operation in $R$, $J$ contains a nonzero ideal $K$ of $R$ such that $K*K=0$.

**Proof.** Let $a$ be a nonzero element in $J$. If $(a)*(a)\neq 0$, there exists a nonzero element $a_1\in (a)*(a)\subseteq (a)\subseteq J$. If $(a_1)*(a_1)\neq 0$, then by Theorem 2 we can continue this to obtain a nonzero element $a_{n+1}\in (a_n)*(a_n)\subseteq J$ such that $(a_{n+1})*(a_{n+1})=0$.

If $R$ is an $s$-ring, there exists a $*$-operation in $R$ such that $A*A=A'$ for every $A\in I(R')$. Hence we have

**Corollary 4.** Each nonzero ideal $J$ of an $s$-ring $R$ which is contained in the prime radical $P(R)$ contains a nonzero nilpotent ideal $K$ of $R$ such that $K^2=0$.

If $R$ is a Lie, alternative or $(-1, 1)$ ring (a 2-ring) then each nonzero ideal of $R$ which is contained in the prime radical contains a nonzero ideal $K$ of $R$ such that $K^2=0$. This improves the result of Fisher [2] for associative rings, which requires the additional assumption that the ring has an identity.

**Definition 2.** An ideal $K$ of $R$ is said to be essentially nilpotent if $K$ contains a nilpotent ideal $L$ of $R$ which is essential in $K$, i.e., $L$ has nonzero intersection with nonzero ideal of $R$ contained in $K$.

Note that every nonzero nilpotent ideal of $R$ is essentially nilpotent. While it is well-known that the prime radical $P(R)$ of an $s$-ring $R$ contains all nilpotent ideals of $R$, it is not known whether $P(R)$ is nilpotent even under the chain condition on one-sided ideals. However we can show that $P(R)$ is essentially nilpotent. This has been proved for associative rings [2].

**Theorem 5.** Let $R$ be an $s$-ring. Every nonzero ideal $J$ of $R$ which is contained in the prime radical $P(R)$ of $R$ is essentially nilpotent.

**Proof.** The proof proceeds as in [2]. Let $\{N_t|t\in T\}$ be the collection of all nonzero nilpotent ideals $N_t$ of $R$ such that $N_t\subseteq J$ and $N_t^2=0$. By Corollary 4 this collection is not empty. Let $Q=\{S\subseteq T|\sum_{t\in S}N_t$ is direct\}. Then $Q$ is non-empty and inductive. Hence by Zorn's lemma one finds a maximal element $U$ in $Q$. Let $N=\sum_{t\in U}N_t$. Since the sum is direct and each $N_t$ is an ideal, we have that $N^2=0$. We show that $N$ is essential in $J$. If not, then there exists a nonzero ideal $K\subseteq J$ of $R$ such that $N\cap K=0$. Corollary 4 then ensures that there exists a nonzero $N_t\subseteq K$ for some $t\in T$ such that...
Hence $N + N_t$ is direct and this contradicts the maximality of $U$. Therefore, $N$ is essential in $J$ and $J$ is essentially nilpotent.

References


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