PROPERTIES OF ALMOST C-CONTINUOUS FUNCTIONS

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1. Introduction

In 1970, Gentry and Hoyle [1] have introduced the concept of c-continuous functions which has been investigated by Long and Hendrix [4] and Long and Herrington [5]. In 1975, Long and Hamlett [3] have defined and studied the concept of H-continuity analogous to that of c-continuity. On the other hand, in 1968 Singal and Singal [8] have introduced a weak form of continuity called almost-continuity. Quite recently, Suk Geun Hwang [9] has introduced a new class of functions, called almost c-continuous functions, which contains the class of c-continuous functions and that of almost-continuous functions. The purpose of the present paper is to continue the investigation of almost c-continuous functions.

2. Preliminaries

Throughout the present paper spaces mean always topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. A subset A is said to be regular closed (regular open) if Cl(Int(A)) = A (resp. Int(Cl(A)) = A).

DEFINITION 2.1. A function f: X → Y is said to be c-continuous [1] if for each x ∈ X and each open neighborhood V of f(x) in Y such that Y - V is compact, there exists an open neighborhood U of x in X such that f(U) ⊂ V.

A subset S of a space X is said to be quasi H-closed relative to X [7] (simply quasi H-closed) if for every cover \{V_α | α ∈ V\} of S by open sets of X, there exists a finite subfamily \(V_0\) of \(V\) such that

\[ S \subseteq \bigcup \{ \text{Cl}(V_α) | α ∈ V_0 \}. \]

DEFINITION 2.2. A function f: X → Y is said to be H-continuous [3] if for each x ∈ X and each open neighborhood V of f(x) such that Y - V is quasi H-closed, there exists an open neighborhood U of x such that f(U) ⊂ V.

DEFINITION 2.3. A function f: X → Y is said to be almost-continuous [8] if
for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \), there exists an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq \text{Int}(\text{Cl}(V)) \).

**Definition 2.4.** A function \( f: X \to Y \) is said to be *almost c-continuous* \([9]\) if for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \) such that \( Y - V \) is compact, there exists an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq \text{Int}(\text{Cl}(V)) \).

Almost-continuous functions are almost c-continuous but the converse is not true in general \([9, \text{Example}]\). The following theorem shows the relationships between the functions defined above.

**Theorem 2.5.** The following implications hold and none of these implications can, in general, be reversed:

\[
\text{continuity} \implies H\text{-continuity} \implies c\text{-continuity} \implies \text{almost c-continuity}.
\]

*Proof.* See \([3, \text{Example 3}; \text{Example 4}]\) and \([9, \text{Example}]\).

### 3. Strongly-closed graphs.

Let \( f: X \to Y \) be a function of a space \( X \) into a space \( Y \). The subset \( \{(x, f(x)) \mid x \in X\} \) of the product space \( X \times Y \) is called the graph of \( f \) and usually denoted by \( G(f) \).

**Definition 3.1** The graph \( G(f) \) is said to be *strongly-closed* \([6]\) if for each \( (x, y) \in G(f) \), there exist open sets \( U \subseteq X \) and \( V \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( [U \times \text{Cl}(V)] \cap G(f) = \emptyset \).

The following lemma is a useful characterization of functions with strongly-closed graphs.

**Lemma 3.2** (Long and Herrington \([6]\)). The graph \( G(f) \) is strongly-closed if and only if for each \( (x, y) \in G(f) \), there exist open sets \( U \subseteq X \) and \( V \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap \text{Cl}(V) = \emptyset \).

**Theorem 3.3** If a function \( f: X \to Y \) has a strongly-closed graph, then it is \( H \)-continuous.

*Proof.* Suppose that \( G(f) \) is strongly-closed. Let \( K \) be any quasi-\( H \)-closed set of \( Y \) and \( x \in f^{-1}(K) \). For each \( y \in K \), \( (x, y) \in G(f) \) and hence, by Lemma 3.2, there exist open sets \( U_y(x) \subseteq X \) and \( V(y) \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( f(U_y(x)) \cap \text{Cl}(V(y)) = \emptyset \). Now, the family \( \{V(y) \mid y \in K\} \) is a cover of \( K \) by open sets of \( Y \). Hence there exists a finite subset \( K_0 \) of \( K \) such that \( K \subseteq \cup \{\text{Cl}(V(y)) \mid y \in K_0\} \). Put \( U = \cap \{U_y(x) \mid y \in K_0\} \). Then \( U \) is an open set of \( X \) containing \( x \) and \( U \cap f^{-1}(K) = \emptyset \). This shows that \( f^{-1}(K) \) is a closed set of \( X \). Therefore, it follows from Theorem 1 of \([3]\) that \( f \) is \( H \)-continuous.
Theorem 3.4. If $Y$ is a locally compact Hausdorff space and $f: X \to Y$ is an almost $c$-continuous function, then $G(f)$ is strongly closed.

Proof. Let $(x, y) \in G(f)$. Then $y \neq f(x)$ and hence there exist disjoint open sets $V_1$ and $V_2$ containing $y$ and $f(x)$, respectively. Since $Y$ is locally compact Hausdorff, there exists an open set $V$ of $Y$ such that $y \in V \subseteq \text{Cl}(V) \subseteq V_1$, where $\text{Cl}(V)$ is compact. Since $\text{Cl}(V)$ is regular closed and compact in $Y$, the almost $c$-continuity of $f$ implies that $f^{-1}(\text{Cl}(V))$ is closed in $X$ [9, Theorem 1]. Put $U = X - f^{-1}(\text{Cl}(V))$. Then $U$ is an open set containing $x$ and $f(U) \cap \text{Cl}(V) = \emptyset$. Hence it follows from Lemma 3.2 that $G(f)$ is strongly closed.

As an immediate consequence of Theorem 2.5, Theorem 3.3 and Theorem 3.4 we have

Corollary 3.5. Let $Y$ be a locally compact Hausdorff space. Then for a function $f: X \to Y$, the following are equivalent:

1. $G(f)$ is strongly closed.
2. $f$ is $H$-continuous.
3. $f$ is $c$-continuous.
4. $f$ is almost $c$-continuous.

Theorem 3.6. If $Y$ is a compact (compact Hausdorff) space and $f: X \to Y$ is an almost $c$-continuous function, then $f$ is almost-continuous (resp. continuous).

Proof. Let $F$ be any regular closed set of $Y$. Since $Y$ is compact, $F$ is compact and hence $f^{-1}(F)$ is closed in $X$ [9, Theorem 1]. Therefore, it follows from Theorem 2.2 of [8] that $f$ is almost-continuous. If $Y$ is compact Hausdorff, then it is regular and hence $f$ is continuous.

Corollary 3.7. Let $Y$ be a compact Hausdorff space. Then, for a function $f: X \to Y$, the following are all equivalent:

1. $f$ is continuous.
2. $f$ is almost-continuous.
3. $G(f)$ is strongly closed.
4. $f$ is $H$-continuous.
5. $f$ is $c$-continuous.
6. $f$ is almost $c$-continuous.

Proof. It is known that an almost-continuous function into a Hausdorff space has a strongly closed graph [6, Theorem 1]. Hence this is an immediate consequence of Theorem 3.6.

Theorem 13 of [4] states that if $f: X \to Y$ is an almost-continuous bijection and $Y$ is a Hausdorff space, then $f^{-1}: Y \to X$ is $c$-continuous. We shall show...
that "almost-continuous" in this result can be replaced by "weakly-continuous". A function \( f: X \to Y \) is said to be weakly-continuous [2] if for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \), there exists an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq \text{Cl}(V) \). Every almost-continuous function is weakly-continuous but the converse is not true in general [8, Example 2.3].

**Lemma 3.8 (Levine [2]).** A function \( f: X \to Y \) is weakly-continuous if and only if \( f^{-1}(V) \subseteq \text{Int}[f^{-1}(\text{Cl}(V))] \) for every open set \( V \) of \( Y \).

**Lemma 3.9.** If \( f: X \to Y \) is a weakly-continuous function and \( K \) is a compact set of \( X \), then \( f(K) \) is quasi \( H \)-closed relative to \( Y \).

**Proof.** Let \( \{ V_\alpha \mid \alpha \in \mathcal{V} \} \) be any cover of \( f(K) \) by open sets of \( Y \). By Lemma 3.8, we have \( K \subseteq \bigcup \{ \text{Int}(f^{-1}(\text{Cl}(V_\alpha))) \mid \alpha \in \mathcal{V} \} \). Since \( K \) is compact, there exists a finite subfamily \( \mathcal{V}_0 \) of \( \mathcal{V} \) such that

\[
K \subseteq \bigcup \{ \text{Int}(f^{-1}(\text{Cl}(V_\alpha))) \mid \alpha \in \mathcal{V}_0 \}.
\]

Therefore, we obtain \( f(K) \subseteq \bigcup \{ \text{Cl}(V_\alpha) \mid \alpha \in \mathcal{V}_0 \} \). This shows that \( f(K) \) is quasi \( H \)-closed in \( Y \).

**Theorem 3.10.** If \( Y \) is a Hausdorff space and \( f: X \to Y \) is a weakly-continuous bijection, then \( f^{-1}: Y \to X \) is \( c \)-continuous.

**Proof.** Let \( K \) be any compact set of \( X \). Then by Lemma 3.9 \( f(K) \) is quasi \( H \)-closed relative to \( Y \). Since \( Y \) is Hausdorff, \( (f^{-1})^{-1}(K) = f(K) \) is closed in \( Y \) [7, (2.5), p.161]. Therefore, it follows from Theorem 1 of [1] that \( f \) is \( c \)-continuous.

### 4. Product spaces.

Let \( \{ Y_\alpha \mid \alpha \in \mathcal{V} \} \) be any family of spaces and \( \prod Y_\alpha \) denote the product space. It is known that if \( Y_\alpha \) is locally compact Hausdorff and \( f_\alpha: X \to Y_\alpha \) is \( c \)-continuous for each \( \alpha \in \mathcal{V} \), then a function \( f: X \to \prod Y_\alpha \) defined by \( f(x) = \{ f_\alpha(x) \} \) is \( c \)-continuous [5, Theorem 2.1]. The following theorem is an improvement of this result.

**Theorem 4.1.** If \( Y_\alpha \) is a locally compact Hausdorff space and \( f_\alpha: X \to Y_\alpha \) is an almost \( c \)-continuous function for each \( \alpha \in \mathcal{V} \), then a function \( f: X \to \prod Y_\alpha \), defined by \( f(x) = \{ f_\alpha(x) \} \) for each \( x \in X \), is \( H \)-continuous.

**Proof.** Let \( (x, y) \in G(f) \). Then \( y \neq f(x) \) and there exists \( \beta \in \mathcal{V} \) such that \( y_\beta \neq f_\beta(x) \). Since \( Y_\beta \) is locally compact Hausdorff and \( f_\beta: X \to Y_\beta \) is almost \( c \)-continuous, \( G(f_\beta) \) is strongly-closed by Theorem 3.4. Hence, by Lemma 3.2 there exist open sets \( U \subseteq X \) and \( V_\beta \subseteq Y_\beta \) containing \( x \) and \( y_\beta \), respectively,
such that \( f_\beta(U) \cap \text{Cl}(V_\beta) = \emptyset \). Put \( V = V_\beta \times \prod_{x \neq \beta} Y_x \), then \( V \) is an open set containing \( y \) and \( f(U) \cap \text{Cl}(V) = \emptyset \). Hence, by Lemma 3.2, \( G(f) \) is strongly \( \varepsilon \)-closed. It follows from Theorem 3.3 that \( f \) is \( H \)-continuous.

**Corollary 4.2.** If \( X \) is Hausdorff, \( Y \) is locally compact Hausdorff and \( f: X \to Y \) is almost \( c \)-continuous, then the graph function \( g: X \to X \times Y \), defined by \( g(x) = (x, f(x)) \) for each \( x \in X \), is \( H \)-continuous.

**Proof.** The identity function \( i_X: X \to X \) is continuous and \( X \) is Hausdorff. Hence it follows from Corollary of [6] that \( G(i_X) \) is strongly-closed. The proof is quite similar to that of Theorem 4.1.

**Theorem 4.3.** If \( Y_\alpha \) is a locally compact Hausdorff space and \( f_\alpha: X_\alpha \to Y_\alpha \) is an almost \( c \)-continuous function for each \( \alpha \in \mathcal{V} \), then a function \( f: \prod_{\alpha \in \mathcal{V}} X_\alpha \to \prod_{\alpha \in \mathcal{V}} Y_\alpha \), defined by \( f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\} \) for each \( \{x_\alpha\} \in \prod_{\alpha \in \mathcal{V}} X_\alpha \), is \( H \)-continuous.

**Proof.** Let \( (x, y) \in G(f) \). Then \( y \neq f(x) \) and there exists \( \beta \in \mathcal{V} \) such that \( y_\beta \neq f_\beta(x_\beta) \). Since \( Y_\beta \) is locally compact Hausdorff and \( f_\beta \) is almost \( c \)-continuous, \( G(f_\beta) \) is strongly-closed by Theorem 3.4. Hence, by Lemma 3.2, there exist open sets \( U_\beta \subseteq X_\beta \) and \( V_\beta \subseteq Y_\beta \) containing \( x_\beta \) and \( y_\beta \), respectively, such that \( f_\beta(U_\beta) \cap \text{Cl}(V_\beta) = \emptyset \). Put

\[
U = U_\beta \times \prod_{x \neq \beta} X_x \quad \text{and} \quad V = V_\beta \times \prod_{x \neq \beta} Y_x.
\]

Then \( U \) and \( V \) are open sets containing \( x \) and \( y \), respectively, such that \( f(U) \cap \text{Cl}(V) = \emptyset \). This shows that \( G(f) \) is strongly-closed. Therefore, it follows from Theorem 3.3 that \( f \) is \( H \)-continuous.

### 5. Compact spaces.

For any space \((Y, \sigma)\) the family \( \mathcal{B} \) of regular open sets having compact complements forms a base for a new topology \( \sigma^* \) on \( Y \). The reason is that if \( U \) and \( \mathcal{V} \) belong to \( \mathcal{B} \), then \( U \cap \mathcal{V} \) is regular open and \( Y - (U \cap \mathcal{V}) = (Y - U) \cup (Y - \mathcal{V}) \) is compact. Let \( f: X \to (Y, \sigma) \) be a function and \( f^*: X \to (Y, \sigma^*) \) a function defined by \( f^*(x) = f(x) \) for each \( x \in X \). Then, it is obvious that \( f \) is almost \( c \)-continuous if and only if \( f^* \) is continuous. Since \( \sigma^* \subseteq \sigma \), the identity function \( i: (Y, \sigma) \to (Y, \sigma^*) \) is continuous and also \( i^{-1}: (Y, \sigma^*) \to (Y, \sigma) \) is almost \( c \)-continuous.

**Theorem 5.1.** For any space \((Y, \sigma)\), the space \((Y, \sigma^*)\) is compact.

**Proof.** Let \( \{V_\alpha | \alpha \in \mathcal{V}\} \) be any \( \sigma^* \)-open cover of \( Y \). Let \( y \in Y \). Then there exist an \( \alpha_0 \in \mathcal{V} \) and \( \mathcal{V} \in \mathcal{B} \) such that \( y \in V \subseteq V_{\alpha_0} \). Since \( Y - \mathcal{V} \) is compact in \((Y, \sigma)\), there exists a finite subfamily \( \mathcal{F}_0 \) of \( \mathcal{V} \) such that \( Y - \mathcal{V} \subseteq \bigcup \{V_\alpha | \alpha \in \mathcal{F}_0\} \).
Therefore, we have $Y = V_{e_0} \cup \bigcup \{ V_{\alpha} | \alpha \in V_0 \}$. This shows that $(Y, \sigma^*)$ is compact.

**Theorem 5.2.** If $(Y, \sigma)$ is a compact Hausdorff space, then $(Y, \sigma^*)$ is Hausdorff and $\sigma^* = \sigma$.

**Proof.** Let $y_1$ and $y_2$ be a pair of distinct points of $Y$. Since $(Y, \sigma)$ is Hausdorff, there exist disjoint $\sigma$-open sets $V_1$ and $V_2$ containing $y_1$ and $y_2$, respectively. Therefore, we have $\text{Int}(\text{Cl}(V_1)) \cap \text{Int}(\text{Cl}(V_2)) = \emptyset$ and $y_j \in V_j \subseteq \text{Int}(\text{Cl}(V_j))$, where $j = 1, 2$. Since $(Y, \sigma)$ is compact, $Y - \text{Int}(\text{Cl}(V_j))$ is compact and $\text{Int}(\text{Cl}(V_j)) \in \sigma^*$. This shows that $(Y, \sigma^*)$ is Hausdorff. Since compact Hausdorff spaces are minimal Hausdorff, we have $\sigma \subseteq \sigma^*$ and hence $\sigma = \sigma^*$.

**Theorem 5.3.** If $(Y, \sigma^*)$ is Hausdorff, then $(Y, \sigma)$ is compact and $\sigma^* = \sigma$.

**Proof.** Let $\{ V_\alpha | \alpha \in \mathcal{P} \}$ be any $\sigma$-open cover of $Y$. Since $(Y, \sigma^*)$ is Hausdorff, there exist disjoint $\sigma^*$-open sets $V_1$ and $V_2$ such that $Y - V_j$ is compact in $(Y, \sigma)$ for $j = 1, 2$. Hence there exists a finite subfamily $\mathcal{P}_j$ of $\mathcal{V}$ such that $Y - V_j \subseteq \bigcup \{ V_\alpha | \alpha \in \mathcal{P}_j \}$, where $j = 1, 2$.

Therefore, we obtain

$$Y = (Y - V_1) \cup (Y - V_2) = \bigcup \{ V_\alpha | \alpha \in \mathcal{P}_1 \cup \mathcal{P}_2 \}.$$  

Hence $(Y, \sigma)$ is compact. Since $\sigma^* \subseteq \sigma$ and $(Y, \sigma^*)$ is Hausdorff, $(Y, \sigma)$ is Hausdorff and hence minimal Hausdorff. Therefore, we obtain $\sigma^* = \sigma$.

**Corollary 5.4.** A space $(Y, \sigma)$ is compact Hausdorff if and only if the space $(Y, \sigma^*)$ is compact Hausdorff.

**Proof.** This is an immediate consequence of Theorem 5.2 and Theorem 5.3.

**Corollary 5.5.** If a function $f : X \to (Y, \sigma)$ is almost $c$-continuous and $(Y, \sigma^*)$ is Hausdorff, then $f$ is continuous.

**Proof.** Since $(Y, \sigma^*)$ is Hausdorff, $(Y, \sigma)$ is compact Hausdorff by Theorem 5.3. Hence it follows from Theorem 3.6 that $f$ is continuous.

**References**

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