ON SEMIDIRECT FACTORS OF A LOCALLY COMPACT GROUP

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1. It is well known that a connected locally compact abelian group $G$ is isomorphic with $\mathbb{R}^n \times E$, where $\mathbb{R}^n$ is a $n$-dimensional vector group and $E$ a connected compact group. This means, in a sense, that the complimentary part of the maximal compact subgroup is determined uniquely.

Now suppose that a locally compact group $G$ is a semidirect product of a connected subgroup $N$ and by a compact subgroup $C$. Is the complimentary part $N$ unique in the sense that, if $G=N'C$ is another way of decomposition as above, $N$ and $N'$ are isomorphic (as topological groups)? We shall show that this is the case if $N$ is, in addition, nilpotent. However, we will give an example that $N$ need not be unique even if it is solvable.

2. As usual, $\mathbb{C}$ and $\mathbb{R}$ will denote the set of complex numbers and real numbers, respectively. Let $T=\{t \in \mathbb{C} : |t|=1\}$ be a circle and $H=\mathbb{C} \times \mathbb{C} \times \mathbb{R}$. Now, let $\eta : T \times T \rightarrow \text{Aut}(H)$ be a continuous homomorphism defined by $\eta(t_1 t_2) (z_1, z_2, r) = (t_1 z_1, t_1 z_2, r)$ and let $G$ be the semidirect product of $H$ by $T \times T$ determined by $\eta$. For each homomorphism $f : \mathbb{R} \rightarrow T \times T$, a subgroup $N_f$ of $G$ will be defined by $N_f = \{z_1, z_2, r, f(r) | z_i \in \mathbb{C}, i=1,2, r \in \mathbb{R}\}$. Since $N_f$ is an extension of commutative group by a commutative group, it is solvable. Therefore, subgroups $N_1 = \{z_1, z_2, r, e^{2 \pi i r}, 1\}$ and $N_2 = \{z_1, z_2, r, e^{2 \pi i r}, e^{-2 \pi i r}\}$, which corresponds to homomorphisms $f_1 : r \rightarrow (e^{2 \pi i r}, 1)$ and $f_2 : r \rightarrow (e^{2 \pi i r}, e^{-2 \pi i r})$, respectively, are solvable. Direct computation shows that the typical elements of $[N_1, N_1]$ and $[N_2, N_2]$ are, respectively, of the form

\[(z_1(1-e^{2\pi i r}) + z_1'(e^{2\pi i r} - 1), 0, 0, 1, 1) \text{ and} \]
\[(z_1(1-e^{-2\pi i r}) + z_1'(e^{2\pi i r} - 1), z_2(1-e^{2\pi i r}) + z_2'(e^{-2\pi i r} - 1), 0, 1, 1).\]

Therefore, $[N_1, N_1] = \mathbb{C}$ and $[N_2, N_2] = \mathbb{C} \times \mathbb{C}$. This means $N_1$ and $N_2$ are not isomorphic, whereas $G=N_1 \cdot H, \ N_1 \cap H = \{e\}$ and $H = \{(0, 0, 0, t_1, t_2) : (t_1, t_2) \in T \times T\}$.

3. We begin with the following

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**Lemma.** Let $N$ be a connected nilpotent locally compact group. Then any compact subgroup $C$ of $N$ is central in $N$.

**Proof.** Let $U$ be any neighborhood of identity element $e$ of $N$ and $K$ a compact normal subgroup of $N$ such that $N/K$ is an analytic group. For the moment assume that any compact subgroup of a nilpotent analytic group is central. Now, $CK/K$ is central and hence we have $[C,N] \subseteq [CK,N] \subseteq K \subseteq U$. Since $U$ is arbitrary, $[C,N] = \{e\}$. Therefore the problem is reduced to the case $N$ is a analytic group. It is well known that the quotient group of an analytic group $N$ modulo its center $Z$ is simply connected. Therefore $CZ/Z$, a compact subgroup of a simply connected nilpotent analytic group, is trivial. Hence $C$ is a central subgroup.

**Theorem.** Let $G$ be a locally compact group and $G = N_1 \cdot C (i=1,2)$ be a semidirect product of a connected nilpotent normal subgroup $N_i$ by a compact subgroup $C$. Then there exist a isomorphism $h : G \to G$ which fixes every point of $C$. In particular, $N_1$ and $N_2$ are isomorphic.

**Proof.** Let $N$ be the nilradical of $G$. There are uniquely determined continuous projections $f_i$ and $g_i$ such that $x = f_i(x)g_i(x)$ ($i=1,2$). Define a continuous map $h : G \to G$ by setting $h(x) = f_2(f_1(x)) \cdot g_1(x)$. We shall show that the map $h$ is a homomorphism of $G$. The following equalities are an easy consequence of the definitions:

$$h(xy) = f_2(f_1(xy))g_1(xy)$$
$$= f_2(f_1(x))g_1(x)f_1(y)g_1(x)^{-1}g_1(xy)$$
$$= f_2(f_1(x))g_2(f_1(x))f_2(g_1(x))f_1(y)g_1(x)^{-1}g_2(f_1(x)^{-1}g_1(xy)).$$

Since $f_1(x) = f_2(f_1(x))g_2(f_1(x))$, $f_2(f_1(x))^{-1}f_1(x) = g_2(f_1(x)) \in N \cap K$ is a central element of $N$. Since $f_2(g_1(x)f_1(y)g_1(x)^{-1}) \in N$, we have

$$h(xy) = f_2(f_1(x))f_2(g_1(x)f_1(y)g_1(x)^{-1})g_1(xy).$$

On the other hand,

$$g_1(x)f_1(y)g_1(x)^{-1} = g_1(x)(f_2(f_1(y))g_2(f_1(y))g_1(x)^{-1}$$
$$= g_1(x)f_2(f_1(y))g_1(x)^{-1}(g_1(x)g_2(f_1(y))g_1(x)^{-1})$$

and hence $f_2(g_1(x)f_1(y)g_1(x)^{-1} = g_1(x)f_2(f_1(y))g_1(x)^{-1}$. Consequently, by (1) we have $h(xy) = h(x)h(y)$. Similarly, the map $h'$ defined by $h'(x) = f_1(f_2(x))g_2(x)$ is a continuous homomorphism which is the inverse of $h$. To show this, let $x, y \in G$. To make the computation simpler, we note that $f_1(f_2(x)) = f_1(x)$ and $f_2(f_1(x)) = f_2(x)$ which is a consequence of the uniqueness of representation of an element as a product.
Now we have

\[ h'(h(x)) = h'(f_2(x)g_1(x)) = f_1(f_2(x)g_1(x))g_2(f_2(x)g_1(x)) \]
\[ = f_1(f_2(x)g_1(x))g_2(g_1(x)) \]
\[ = f_1(f_2(x)g_1(x))g_1(x) \]
\[ = f_1(f_1(f_2(x))g_1(f_2(x)g_1(x)))g_1(x) \]
\[ = f_1(x)g_1(x) = x. \]

This implies that \( h' \) is the inverse of \( h \) and hence that \( h \) is a homeomorphism.

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