NOTE ON THE SEMIGROUP OF FUZZY MATRICES

BY JIN BAI KIM

1. Introduction

Let $F$ be a finite subset of the unit interval $[0,1]$ of the real line. $M_n(F)$ denotes the set of all $n \times n$ fuzzy relation matrices over $F$. Then $M_n(F)$ form a semigroup under the matrix multiplication and we call it the semigroup of fuzzy (relation) matrices over $F$ [5]. We denote by $C_n(F)$ the set of all circulant fuzzy (relation) matrices over $F$. Then $C_n(F)$ is a commutative subsemigroup of $M_n(F)$ [6]. [5, 6, 7] are main references for $M_n(F)$ and $C_n(F)$. We define $kIr = (a_{ij})$ by $a_{ij} = k$ if $i = j = 1, 2, \ldots, r$ and $a_{ij} = 0$ if $i \neq j$ or if $i = j = r + 1, r + 2, \ldots, n$. Let $R^n_{rk}$ and $D^n_{rk}$ be respectively an $R$-class and a $D$-class containing $kIr \in M_n(F)$, where $k$ is a member of $F$. We find the cardinal numbers $|R^n_{rk}|$ and $|D^n_{rk}|$ of the sets $R^n_{rk}$ and $D^n_{rk}$. We find the number of all idempotents in $D^n_{rk}$.

2. Elementary properties of the semigroup

We list some elementary properties of $M_n(F)$ and $C_n(F)$.

(1) $M_n(F)$ is not regular. $C_n(F)$ is an abelian subsemigroup of $M_n(F)$ with the cardinality $|C_n(F)| = (|F|)^n$.

(2) If $A \in C_n(F)$, then $A^h$ is an idempotent for some $h \leq n$.

(3) Any idempotent $A$ in $C_n(F)$ is of the form

$$A = a_1E + a_2(P^{d_1} + P^{2d_1} + \cdots + P^{(r-1)d_1}) + \cdots + a_m(P^{d_m} + P^{2d_m} + \cdots + P^{(r-1)d_m}),$$

where $n = d_1 + d_2 + \cdots + d_m$, $d_1 \neq d_2 \neq \cdots \neq d_m$, $a_1 \neq 0$, and $E$ is the identity of $M_n(F)$. (See [5, 6] for proofs of items (2) and (3).

(4) Let $r(A)$ and $c(A)$ denote respectively the row-rank and column-rank of $A \in M_n(F)$. Then $r(A) = 1$ iff $c(A) = 1$. If $D_1 = \{X \in M_n(F) : r(X) = 1\}$, then $D_1$ is the union of $k$ $D$-classes of $M_n(F)$, where $k = |F| - 1$. For simplicity, we write $F = \{0, 1, 2, \ldots, k\}$.

We denote by $D^n_{ra}$ the $D$-class of $M_n(F)$ containing $aI_r$, $0 \neq a \in F$. Received Dec. 15, 1978
THEOREM 1. Let \( 0 \neq a \neq b \neq 0 \). Then \( D^*_r \cap D^*_r = \emptyset \) (the empty set). Let \( D^*_r \) be the union of all \( D^*_r \) \((0 \neq a \in F)\). If \( A \in D^*_r \), then \( r(A) = c(A) \).

Proof. Assume that \( a > b \). There is no \( X \) in \( M_n(F) \) such that \( aI_r = (bI_r)X \) and consequently we have shown that \( D^*_r \cap D^*_r = \emptyset \). We show that if \( A \in D^*_r \), then \( r(A) = r = c(A) \). To show that let \( A \in D^*_r \). Then there exists \( B \) such that \( A \perp B \) and \( B \perp (aI_r) \). There exist \( X, Y, U, V \) such that \( BX = aI_r \), \( UA = B \), \( aI_r Y = B \) and \( VB = A \). We obtain \( C(aI_r) \subseteq C(B) \subseteq C(aI_r) \) and hence \( C(aI_r) = C(B) \), where \( C(B) \) denotes the column space of \( B \). Thus \( c(B) = c(aI_r) = r = c(A) \). Since \( B \) has at most \( r \) non-zero rows, we have that \( r(B) \leq r \). From \( c(B) = r \), it follows that there exists a submatrix \( G = (c_{ij}) \) of \( B \) of order \( r \) such that \( c_{ii} = a \) \((i = 1, 2, \ldots, r)\) and \( c_{ij} = 0 \) for \( i \neq j \). This means that \( r(B) = r \). Now from \( UA = B \) and \( VB = A \), we have that \( R(A) = R(VB) \subseteq R(B) = R(UA) \subseteq R(A) \) and \( R(A) = R(B) \), where \( R(B) \) denotes the row space of \( B \). We obtain \( r(A) = r(B) = r \). Similarly, we can show that \( c(A) = c(B) = r \). This proves Theorem 1.

3. The \( R \)-class \( R^*_r \)

We note that \( F = \{0, 1, 2, \ldots, k\} \). Let \( 0 \neq m \in F \). \( R^*_r \) denotes the \( R \)-class containing \( mI_r \). We compute the cardinality \( |R^*_r| \) of the set \( R^*_r \) in the following theorem. \( \binom{n}{k} \) denotes the binomial coefficient.

THEOREM 2. Let \( r \) be a positive integer with \( 1 \leq r \leq n \). Then

\[
|R^*_r| = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (m+1)^{r-i} n.
\]

To prove the theorem 2 we need lemmas.

**LEMMA 1.** \( \binom{t}{m} + \binom{t}{m+1} = \binom{t+1}{m+1} \).

**LEMMA 2.** \( |R^*_1| = (m+1)^n - m^n \).

Proof. We supply two proofs of Lemma 2. \( V_n(m) \) denotes the set of all \( 1 \times n \) matrices over \( F = \{0, 1, 2, \ldots, m\} \). Let \( \emptyset \) denote the \((n-1) \times n\) zero matrix. Letting \( x = (x_1, x_2, \ldots, x_n) \in V_n(m) \), \( \bar{x} = \begin{pmatrix} x \end{pmatrix} \) denotes an \( n \times n \) matrix formed from \( x \) and \( \emptyset \). Then \( \pi \in R^*_1 \) iff \( x \) contains at least one \( m \) as its component. We see that \( |V_n(m)| = (m+1)^n \). Therefore we see that

\[
|R^*_1| = |V_n(m)| - |V_n(m-1)| = (m+1)^n - m^n,
\]

proving the lemma.

The second proof is given by the following expression.
Note on the semigroup of fuzzy matrices

\[ |R_{1m}^n| = \frac{n}{1} \left( \sum_{i=0}^{n-1} \binom{n-1}{t} (m-1)^t \right) + \frac{n}{2} \left( \sum_{i=0}^{n-2} \binom{n-2}{t} (m-1)^t \right) + \ldots + \binom{n}{n} (m-1)^t + \ldots + \binom{n}{m} = (m+1)^n - m^n. \]

A term of \(|R_{1m}^n|\) with the coefficient \(\frac{n}{i}\) expresses the number of \(x = \left( \frac{x}{0} \right)\) of \(R_{1m}^n\) such that \(\{j: x_j = m\} = i\), where \(x = (x_1, x_2, \ldots, x_n) \in V_n(m)\). This also proves the lemma.

**Lemma 3.** Let \(F = \{0, 1, 2, \ldots, k\}\). Let \(V_r(F)\) denote the row vector space over \(F\). Then \(|V_r(F)| = (k+1)^r\).

**Proof of Theorem 2.** We prove this theorem by induction on \(r\). We proved this theorem for case \(r=1\) in Lemma 2. We assume that Theorem 2 is proved for all \(r\) less than \(r_0\), where \(r_0\) being a fixed positive integer less than \(n\) and greater than 1.

Consider \(|R_{r_0m}^n|\). By induction assumption we have that

\[ |R_{r_0-1m}^n| = \sum_{i=0}^r (-1)^i \binom{r_0-1}{t} ((m+1)^r_0 - i)^n = Q. \]

The first term of \(Q\) is \((m+1)^{r_0-1}n\) which becomes

\[ ((m+1)^{r_0})^n - ((m+1)^{r_0-1}n) = (u)^n - (u-1)^n \]

if we assume \(R_{r_0-1m}^n\) has changed to \(R_{r_0m}^n\), by a careful application of Lemma 4, where \(u = (m+1)^{r_0}\) which is justified by Lemma 3. Consider the second term \(-(r_0-1) ((m+1)^{r_0-1}n) of Q. This quantity becomes \(- (r_0-1) (v^n - (v-1)^n)\) as \(R_{r_0-1m}^n\) becomes \(R_{r_0m}^n\) by Lemma 4, where \(v = (m+1)^{r_0-1}\) which is justified by Lemma 3. In similar fashion, by applying Lemma 4 and Lemma 3 to each term \(-1)^i \binom{r_0-1}{t} ((m+1)^{r_0-1} - i)^n = q\) of \(Q\), we claim that \(q\) becomes

\[ (-1)^i \binom{r_0-1}{t} ((m+1)^{r_0} - i)^n - ((m+1)^{r_0-1} - (i+1))^n \]

as \(R_{r_0-1m}^n\) becomes \(R_{r_0m}^n\). Now using the identity \(\binom{t}{s-1} + \binom{t}{s} = \binom{t+1}{s}\) of Lemma 1, we obtain

\[ |R_{r_0m}^n| = (m+1)^{r_0} - \binom{r_0}{1} ((m+1)^{r_0-1}n + \ldots + (-1)^{r_0} \binom{r_0}{r_0} ((m+1)^{r_0} - r_0)^n \]

\[ = \sum_{i=0}^r (-1)^i \binom{r_0}{i} ((m+1)^{r_0} - i)^n. \]
This proves Theorem 2.

**Theorem 3.** \(|D_{rm}^n| = \frac{1}{r!} (|R_{rm}^n|)^2\).

**Proof.** We can prove that \(|L_{rm}^n| = |R_{rm}^n|\), where \(L_{rm}^n\) denotes the \(L\)-class containing \(mI_r\). We can also show that \(|H_{rm}^n| = r!\), where \(H_{rm}^n\) denotes the \(H\)-class containing \(mI_r\). This proves Theorem 3.

4. The number of all idempotents in \(D_{rm}^n\)

\(E(S)\) denotes the set of all idempotents of a subset \(S\) of a semigroup. We find \(|E(D_{rm}^n)|\) in Theorem 4.

**Lemma 4.** \(|E(D_{1m}^n)| = ((m+1)^2)^n - ((m+1)^2 - 1)^n\).

We give two different proofs of Lemma 4.

**Proof.** (1). Let \(x = (x_1, x_2, \ldots, x_n) \in V_n(F)\) and \(0\) denotes the \((n-1) \times n\) zero matrix. Then \(\begin{pmatrix} x \\ 0 \end{pmatrix} \in R_{1m}^n\) if \(x_i = m\) for some \(i\). \(\begin{pmatrix} x \\ 0 \end{pmatrix} \in R_{1m}^n\) is an idem­

totent iff \(x_1 = m\). Thus the number of all idempotents in \(R_{1m}^n\) is equal to \((m+1)^n - 1\). It is not difficult to show that there are \(u\) \(R\)-classes each of which contains exactly \((m+1)^n - 1\) idempotents, where \(u = \binom{n}{2} (m^2)\). Now consider an \(R\)-class \(R_A\) which contains \(A = (a_{ij})\), where \(a_{11} = a_{21} = m\) and \(a_{ij} = 0\) if \(a_{11} \neq a_{ij} \neq a_{21}\). We can show that \(R_A \subseteq D_{1m}^n\) and \(|E(R_A)| = (m+1)^{n-2} ((m+1)^2 - m^2)\). We can show that there are exactly \(u\) \(R\)-classes \(R_B\) such that \(|E(R_A)| = |E(R_B)|\) and \(R_B \subseteq D_{1m}^n\), where \(u = \binom{n}{2} m^{n-2}\). By the foregoing argument, we can write

\[|E(D_{1m}^n)| = \binom{n}{1} (m^{n-1} (m+1)^{n-1}) + \binom{n}{2} (m^{n-2} (m+1)^{n-2} ((m+1)^2 - m^2)) + \]
\[\cdots + \binom{n}{r} m^{n-r} (m+1)^{n-r} ((m+1)^2 - m^r) + \cdots + \binom{n}{n} ((m+1)^n - m^n)\]
\[= (m+1)^{2n} - ((m+1)^2 - 1)^n.\]

This proves Lemma 4.

(2) (The second proof of Lemma 4). For simplicity we write \(F = \{0, 1, 2, \ldots, m\}\) instead of \(F = \{0 = r_1, r_2, \ldots, r_m = 1 : r_i \in [0, 1]\}\). Consider \(E(D_{1m}^n)\). \(A \in E(D_{1m}^n)\) iff there exist \(x = (x_1, x_2, \ldots, x_n)^t\) (\(t\) means that ‘transpose’) and \(y = (y_1, y_2, \ldots, y_n)\) such that \(xy = A\) and \(yx = m\). This shows that if \(A \in\)
Note on the semigroup of fuzzy matrices

Let $E(D^n_{1m})$ then there exists $i$ such that $x_i = m = y_i$ and $xy = A$. Now we consider the meaning of $(m+1)^{2n} - m^{2n}$. $(m+1)^{2n}$ means the number of $xy = A$ such that $x = (x_1, x_2, \ldots, x_n)^t$ and $y = (y_1, y_2, \ldots, y_n), x_i, y_i \in F$. $m^{2n}$ means that

$$| \{ xy : x_i, y_j \in \{0, 1, 2, \ldots, m-1\} = F \setminus m \} | = m^{2n}. $$

This proves that $| E(D^n_{1m}) | = (m+1)^{2n} - m^{2n}$.

Note that if we set $r=1$, in Theorem 4, then $2(m+1)^r + rm^2 - 1 = (m+1)^2$. If $r=2$, then $2(m+1)^r + rm^2 - 1 = (2m+1)^2$.

Definition. We denote by $M_{mn}(F)$ the set of all $m \times n$ matrices over $F$. We define a square matrix $I(m, r) =$ $(a_{ij})$ of order $r$ as the following:

$$a_{ii} = m \quad \text{and} \quad a_{ij} = 0 \quad \text{for all} \quad i \neq j. $$

We define

$$X = \{ x = (I(m, r) : x) \in M_{r+1, r}(F) \} \quad \text{and} \quad Y = \{ y = (I(m, r), y) \in M_{r+1, r+1}(F) \}, $$

where $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)^t$. Note that $x y$ is a member of $M_{r+1, r+1}(F)$ for $x \in X$ and $y \in Y$.

Lemma 5. Let $x y = \{ x y : x \in X, \ y \in Y \}$. Then

1. $| E(x y) | = (m+1)^r$ when $x = (0, 0, \ldots, 0)$.
2. $| E(x y) | = m+1$ when $x = (x_1, x_2, \ldots, x_r)$ contains just one nonzero element.
3. $| E(x y) | = 1$ when $x$ contains at least two nonzero components.

The proof of Lemma 5 is trivial and we omit it.

Lemma 5 has an important meaning in Theorem 4. For (2) of Lemma 5, there exist $mr$ such vectors $x = (x_1, x_2, \ldots, x_r)$ each of which contains just one non-zero element (referring to $F = \{0, 1, 2, \ldots, m\}$). We define a number $rm(m+1)$ for (2). Consider (1) of Lemma 5. There is just one vector $x = (0, 0, \ldots, 0)$, the zero-vector, and we define a number $(m+1)^r$ for (1). For (3), we define a number $(m+1)^r - mr - 1$. The sum of these three numbers $rm(m+1)$, $(m+1)^r$ and $(m+1)^r - mr - 1$ is equal to $2(m+1)^r + r m^2 - 1 = t$ and $t$ has a significant meaning for $| E(D^n_{1m}) |$.

Theorem 4. $| E(D^n_{1m}) | = \frac{1}{r!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} (t-i)^n$,

where $t = 2(m+1)^r + r m^2 - 1$.

Proof. We prove this theorem by induction on $r$. For the case of $r=1$, we have Lemma 4 which proves Theorem 4. We assume that the theorem is proved for all $r < r_0$, where $r_0$ is a fixed positive integer less than $n$. This means that
\[ |E(D^*_{r_0-1,m})| = \frac{1}{(r_0-1)!} \sum_{i=0}^{r_0-1} (-1)^i \binom{r_0-1}{i} (t(r_0-1)-i)^n, \]

where \( t(s) = 2(m+1)^s + sm^2 - 1 \). Consider \( A \) in \( D^*_{r_0m} \). For \( A \) there exist \( U \in L^*_{r_0m} \) and \( V \in R^*_{r_0m} \) such that \( A = UV \). For a moment, we suppose that \( A = \overline{A} = \overline{U} \overline{V} \), \( \overline{U} \in L^*_{r_0-1,m} \), \( \overline{V} \in R^*_{r_0-1,m} \). Applying Lemma 4 to \( \overline{A} \) which changes to \( A \), and a careful consideration of the meaning of Lemma 5, we can realize that the first term \( \binom{r_0-1}{0} (t(r_0-1))^n \) of \( |E(D^*_{r_0-1,m})| \) changes to \( \binom{r_0-1}{0} (t(r_0))^n - (t(r_0)-1)^n \) as \( \overline{A} \) changes to \( A \) or \( r_0-1 \) changes to \( r_0 \). The foregoing argument applies to the second term \( \binom{r_0-1}{1} (t(r_0-1)-1)^n \) which changes to \( \binom{r_0-1}{1} (t(r_0)-1)^n - (t(r_0)-2)^n \) as \( \overline{A} \) changes to \( A \). We apply the foregoing argument to each term of \( |E(D^*_{r_0-1,m})| \) and using \( \binom{s}{m} + \binom{s}{m-1} = \binom{s+1}{m} \) of Lemma 1, we obtain the formula for \( |E(D^*_{r_0m})| \). This proves the theorem.

5. Some additional results

It is difficult to find the number of all \( D \)-classes of the semigroup \( M_2(F) \) but we have the following.

**Proposition 1.** The number of all \( D \)-classes of the semigroup \( M_2(F) \) of all \( 2 \times 2 \) fuzzy matrices over \( F \) is given by \( \sum_{t=0}^{m} (2t^2 - t + 1)(m-t+1) \), where \( m \) is given by \( |F| = m+1 \).

Note that if \( m=1 \), then the number above is equal to 4. We know that the semigroup \( M_2([0,1]) \) of all \( 2 \times 2 \) boolean matrices over the set \( \{0,1\} \) of two elements has four \( D \)-classes:

\[ D_{(0,0)}, D_{(1,0)}, D_{(0,1)} \text{ and } D_{(1,1)}. \]

We consider now Theorem 2. Let \( A \in M_2(F) \). \( R(A) \) and \( L(A) \) denote respectively the row space and the column space of \( A \). \( R_A \) and \( L_A \) denote respectively the \( R \)-class and the \( L \)-class containing \( A \). The following is the generalized from of Theorem 2.

**Proposition 2.** \( |R_A| = \sum_{t=0}^{e(A)} (-1)^i \binom{e(A)}{i} (|L(A)| - i)^n \) and

\[ |L_A| = \sum_{t=0}^{r(A)} (-1)^i \binom{r(A)}{i} (|R(A)| - i)^n. \]
The proof of Proposition 2 is similar to that of Theorem 2 and we omit the proofs of Propositions 1 and 2.

References


West Virginia University, U. S. A.