In this paper we consider a hypersurface $N$ of a submanifold $\overline{N}$ of the Euclidean space $E^m$. Let $\xi$ be a unit normal vector field on $U \subset N$ in $\overline{N}$. If $S^{m-1}$ is the hypersphere of $E^m$ with centre the origin $(0, \ldots, 0)$ and with radius 1 and if $\xi = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}$, where $x^1, \ldots, x^m$ is the standard coordinate system of $E^m$, then the Gauss map of $N$ in $\overline{N}$ is given by $\eta: U \rightarrow S^{m-1}; p \mapsto (a^1(p), \ldots, a^m(p))$. Let $\bar{\omega}$ (resp. $\omega$) be a volume element of the spherical image of $\overline{N}$ (resp. of $N$) at the point $\eta(p)$ (resp. at the point $p$). In section 3, we look for the connection between $\bar{\omega}$ and $\omega$ in the following cases: (a). the vector field $\xi$ is parallel in the normal bundle $N^\perp$, (b). $N$ is totally geodesic in $E^m$, and (c). $N$ is totally geodesic in $\overline{N}$ and $\xi$ determines at each point an asymptotic direction of $\overline{N}$.

1. Introduction

We shall assume throughout that all manifolds, maps, vector fields, etc. ... are differentiable of class $C^\infty$.

Suppose that $\overline{N}$ is a $(n+1)$-dimensional submanifold of the Euclidean space $E^m$ ($m > n + 1$) and that $N$ is a $n$-dimensional submanifold (hypersurface) of $\overline{N}$. Consider in a neighborhood $U$ of a point $p \in N$ a unit normal vector field $\xi$ on $N$ in $\overline{N}$. The standard Riemann connection of $E^m$ and the Riemann connections of $\overline{N}$ and $N$ are respectively denoted by $\overline{D}$, $\overline{D}$ and $D$.

The Weingarten map $L$ of $N$ in $\overline{N}$ is given by

$$\overline{D}_Y \xi = L(X), \; \forall X \in N_p,$$

and det $L$ is the Gauss curvature at the point $p$ of the hypersurface $N$ of $\overline{N}$. If $Y$ and $Z$ are vector fields of $N$, then we have

$$\overline{D}_Y Z = D_Y Z + V'(Y, Z),$$

where $V'(Y, Z)$ is the second fundamental form of $N$ in $\overline{N}$. Moreover, we find
The metric tensor is denoted by $< , >$

$$\bar{D}_Y Z - <L(Y), Z> \xi.$$  \hfill (1.2)

Let $U$ and $W$ be vector fields of $\bar{N}$, then

$$\bar{D}_U W = \bar{D}_U W + \bar{V}(U, W),$$  \hfill (1.3)

where $\bar{V}(U, W)$ is the second fundamental form of $\bar{N}$ in $E^m$. From (1.2) and (1.3) it follows that

$$\bar{D}_Y Z = \bar{D}_Y Z - <L(Y), Z> \xi + \bar{V}(Y, Z).$$  \hfill (1.4)

But, if $V(Y, Z)$ is the second fundamental form of $\bar{N}$ in $E^m$, then we also have

$$\bar{D}_Y Z = \bar{D}_Y Z + V(Y, Z),$$  \hfill (1.5)

and so, because of (1.4) and (1.5) we find that for each two vector fields $Y$ and $Z$ of $N$

$$V(Y, Z) = -<L(Y), Z> \xi + \bar{V}(Y, Z).$$  \hfill (1.6)

The equation of Weingarten of $N$ in $E^m$, with respect to the unit normal field $\xi$ is given by

$$\bar{D}_X \xi = -(A_\xi(X)) + D_\perp X, \quad \forall X \in N_p,$$  \hfill (1.7)

where $A_\xi$ determines a self adjoint linear map in the tangent spaces of $N$ and $D_\perp$ is a metric connection in the normal bundle $N^\perp$. We also have

$$\bar{D}_X \xi = \bar{D}_X \xi + \bar{V}(X, \xi)$$
or

$$\bar{D}_X \xi = L(X) + \bar{V}(X, \xi).$$  \hfill (1.8)

From (1.7) and (1.8) it follows that

$$L(X) = -(A_\xi(X))$$  \hfill (1.9)
and

$$D_\perp X = \bar{V}(X, \xi), \quad \forall X \in N_p.$$  \hfill (1.10)

Because of (1.9) we have $\det L = \pm \det A_\xi$, which means that the Gauss curvature at the point $p$ of the hypersurface $N$ of $\bar{N}$ is equal to $\pm K(p, \xi_p)$, where $K(p, \xi_p)$ is the Lipschitz–Killing curvature at $p$ of $N$ in $E^m$ with respect to $\xi_p$.

Suppose that $\bar{R}$ is the curvature tensor of $\bar{N}$ and that $U_1, \ldots, U_4$ are $\bar{N}$-vector fields, then the Gauss equation of $\bar{N}$ in $E^m$ is given by
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\begin{align}
\langle U_1, \bar{R}(U_2, U_3) U_4 \rangle &= \langle \bar{V}(U_2, U_1), \bar{V}(U_3, U_4) \rangle \\
&\quad - \langle \bar{V}(U_2, U_4), \bar{V}(U_3, U_1) \rangle.
\end{align}

(1.11)

If \( X \in N_p \), then the Riemann curvature of \( \bar{N} \) at \( p \) in the two-dimensional direction \((X, \xi_p)\) is given by

\[
\bar{K}(X, \xi_p) = \frac{\langle X, \bar{R}(X, \xi_p) \xi_p \rangle}{\langle X, X \rangle},
\]

and because of (1.11),

\[
\langle X, \bar{R}(X, \xi_p) \xi_p \rangle = \langle \bar{V}(X, X), \bar{V}(\xi_p, \xi_p) \rangle - \langle \bar{V}(X, \xi_p), \bar{V}(X, \xi_p) \rangle.
\]

(1.12)

Let \( Y \) and \( Z \) be \( N \)-vector fields, then it follows from (1.11) that \( \langle Y, \bar{R}(Z, \xi) \xi \rangle \) determines a \( 2 \)-covariant symmetric \( N \)-tensor field. Suppose that the principal directions of this tensor field are locally (i.e. in the domain of the unit normal field \( \xi \)) given by the orthonormal base field \( e_1, \ldots, e_n \) of \( N \). Then \( \langle (e_i)_p, \bar{R}((e_i)_p, \xi_p) \xi_p \rangle \) \((i=1, \ldots, n)\) are the extremal values of the Riemann curvatures of \( \bar{N} \) at \( p \) in the two-dimensional directions of \( \bar{N}_p \) which contain \( \xi_p \) (or, in other words, of the Riemann curvatures \( \bar{K}(X, \xi_p) \), \( V X \in N_p \)).

**Definition.** The total normal Riemann curvature of \( \bar{N} \) at the point \( p \in N \) is given by

\[
\bar{K} = \prod_{i=1}^n \langle (e_i)_p, \bar{R}((e_i)_p, \xi_p) \xi_p \rangle.
\]

2. **The Gauss map**

Suppose that \( x^1, \ldots, x^m \) is the standard coordinate system of \( E^m \), with coordinate vector fields \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m} \). \( S^{m-1} \) is the hypersphere of \( E^m \) with centre the origin \((0, \ldots, 0)\) and with radius 1. For the unit normal vector field \( \xi \), with domain \( U \), we have \( \xi = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \) and \( a^i \) \((i=1, \ldots, m)\) are \( C^\infty \) functions over \( U \).

**Definition.** The Gauss map of \( N \) in \( \bar{N} \) is given by

\[
\eta : U \rightarrow S^{m-1} ; \quad p \rightarrow (a^1(p), \ldots, a^m(p)).
\]

Let \( X \in N_p \) and consider a curve \( \sigma : [a, b] \rightarrow N \) such that \( \sigma(a) = p \) and \( T_{\sigma(a)} = X \). Then \( \eta \circ \sigma \) is a curve on \( S^{m-1} \) and we have
\[ \eta_*(X) = T_{\gamma \sigma (\omega)} = \sum_{i=1}^{n} da_i \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x^i} \right) \eta(\rho) \]

- or

\[ \eta_*(X) = \sum_{i=1}^{n} X(a^i) \left( \frac{\partial}{\partial x^i} \right) \eta(\rho). \]  

We also find \( D_X \xi = \sum_{i=1}^{n} a^i \left( \frac{\partial}{\partial x^i} \right) = \sum_{i=1}^{n} X(a^i) \left( \frac{\partial}{\partial x^i} \right) \eta(\rho) \) and thus, because of (1.7) and (1.8),

\[ - (A_\xi(X)) + D_X \xi = L(X) + \tilde{V}(X, \xi) = \sum_{i=1}^{n} X(a^i) \left( \frac{\partial}{\partial x^i} \right) \eta(\rho). \]  

3. a. The vector field \( \xi \) is parallel in the normal bundle \( N^\perp \).

In this case we have \( D_X \xi = 0 \), \( \forall X \in N_\rho \) and \( \forall \rho \in U \), or

\[ L(X) = \sum_{i=1}^{n} X(a^i) \left( \frac{\partial}{\partial x^i} \right) \eta(\rho), \quad \forall X \in N_\rho \]  

The variable point with coordinates \( (a^1(q), \ldots, a^m(q)), \ q \in U \) describes a submanifold \( S \) of \( S^{m-1} \) and \( \dim S = \eta(\rho) \); \( S \) is the spherical image of \( N \) in the neighborhood \( U \) of the point \( \rho \).

We restrict ourselves to the case \( \det L \neq 0 \) at the point \( \rho \).

**Theorem 1.** Suppose the \( \omega \) is a volume element of the spherical image \( S \) at the point \( \eta(\rho) \) and that \( \omega \) is a volume element of \( N \) at the point \( \rho \), then

\[ \eta^*(\omega) = \pm (\det L) \omega. \]

**Proof.** Because of (2.1) and (3.1), we know that the vectors \( \eta_*(X) \) and \( L(X), \ \forall X \in N_\rho \) have the same components with respect to the coordinate bases \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m} \) at the points \( \eta(\rho) \) and \( \rho \). But \( \det L \neq 0 \) at \( \rho \) and therefore \( \eta_* \) is a bijection of \( N_\rho \) on \( S_{\eta(\rho)} \). In this case \( S \) is \( n \)-dimensional (moreover there exists a neighborhood of \( \rho \) in which \( \eta \) is a diffeomorphism). Let \( X_1, \ldots, X_n \) be an orthonormal set of eigenvectors of \( L \) at \( \rho \) and denote the dual forms by \( \omega_1, \ldots, \omega_n \). Then \( \eta_*(X_1), \ldots, \eta_*(X_n) \) form an orthogonal base of \( S_{\eta(\rho)} \). Consider the orthonormal base

\[ \eta_*(X_i) / \langle L(X_i), \ L(X_i) \rangle^{1/2}, \ i = 1, \ldots, n \]

and denote the dual forms by \( \bar{\omega}_1, \ldots, \bar{\omega}_n \). If \( \rho_i \ i = 1, \ldots, n \) are the eigenvalues
of \( L \) at \( p \), then
\[
\langle L(X_i), L(X_i) \rangle^{1/2} = |\rho_i|, \quad i = 1, \ldots, n.
\]
Thus we find \( \eta^*(ω_i) = |\rho_i|ω_i \), \( i = 1, \ldots, n \) and we get
\[
\eta^*(ω_1 \wedge \cdots \wedge ω_n) = \eta^*(ω_1) \wedge \cdots \wedge \eta^*(ω_n) = |\rho_1 \cdots ρ_n|ω_1 \wedge \cdots \wedge ω_n = |\det L|ω_1 \wedge \cdots \wedge ω_n,
\]
which has to be proved.

REMARKS.

1. In a classical way one should formulate the statement of Theorem 1 as follows: if the vector field \( \xi \) is parallel in the normal bundle \( N^\perp \), then the Gauss curvature at the point \( p \) of \( N \) in \( \overline{N} \) or the Lipschitz-Killing curvature \( K(p, \xi(p)) \) of \( N \) is equal to the ratio of volume element of the spherical image of \( N \) and the volume element of \( N \) at \( p \).

2. If \( \det L = 0 \) at \( p \), then \( \dim S < n \) in a neighborhood of \( p \), or \( \dim S = n \), but in this case the function \( \eta^*: F^1(S_{\xi(p)}) → F^1(N_p) \) (\( F^1 \) means the vector space of 1-forms) is no more a bijection and then \( \eta^*(ω) = 0 \). Thus we can say that theorem 1 remains true for \( \det L = 0 \).

3. Suppose that \( \overline{N} \) is a hypersurface of \( E^m \), with unit normal vector field \( τ \) and with Weingarten map \( \overline{L} \), then we have
\[
\overline{V}(X, \xi(p)) = -\langle \overline{L}(X), \xi(p) \rangle \tau_p.
\]
And therefore \( \xi \) is parallel in the normal bundle \( N^\perp \) iff \( \overline{L}(X) \perp \xi(p) \), \( \forall X ∈ N_p \) and \( \forall p ∈ U \), i.e., \( \xi \) determines at each point \( p ∈ U \) a principal direction of the hypersurface \( \overline{N} \).

EXAMPLES

1. Consider any hypersurface \( N \) of the \((n+1)\)-dimensional Euclidean space \( E^{n+1} = \overline{N} ⊂ E^m \) \( (m > n+1) \). Then the (local) normal unit vector field \( \xi \) (or \(-\xi\)) of \( N \) in \( E^{n+1} \) is parallel in the normal bundle \( N^\perp \) and we find the well-known geometric interpretation for the Gauss curvature of a hypersurface of an Euclidean space.

2. Consider in \( E^m \) \( (m > 4) \) the sphere \( N \) with parametric representation
\[
\begin{align*}
x^1 &= a \cos u \cos \upsilon, & x^2 &= a \cos u \sin \upsilon, \\
x^3 &= a \sin u, & x^j &= 0 \quad j = 4, \ldots, m, & a > 0.
\end{align*}
\]
The vector field \( \xi \) with components \( \left( \frac{\cos u \cos \upsilon}{\sqrt{2}} , \frac{\cos u \sin \upsilon}{\sqrt{2}} , \frac{\sin u}{\sqrt{2}} , \frac{1}{\sqrt{2}}, 0, \ldots, 0 \right) \) is clearly a \( C^\infty \) normal unit vector field on \( N \) and it is parallel
in the normal bundle \( N^\perp \). Consider the manifold \( \overline{N} \), represented by

\[
\begin{align*}
\begin{aligned}
x^1 &= a \cos u \cos v + \frac{k}{\sqrt{2}} \cos u \cos v, \\
x^2 &= a \cos u \sin v + \frac{k}{\sqrt{2}} \cos u \sin v, \\
x^3 &= a \sin u + \frac{k}{\sqrt{2}} \sin u, \\
x^4 &= \frac{k}{\sqrt{2}}, \\
x^s &= 0, \quad s = 5, \ldots, m \quad \text{and} \quad k \in \mathbb{R}.
\end{aligned}
\end{align*}
\]

Then \( N \) is a hypersurface of \( \overline{N} \) and it is at once clear that the relative total curvature of \( N \) with respect to \( \xi \) is equal to \( 1/2a^2 \) at each point of \( N \), while it is easy to see that we also have \( \eta^*(\omega) = \pm \omega/2a^2 \).

b. \( N \) is totally geodesic in \( E^m \)

In this case we have \( V(Y, Z) = 0 \), for each two \( N \)-vector fields. From (1.6) it follows that

\[ L = 0 \quad (\text{i.e. } \ N \text{ is totally geodesic in } \overline{N}) \quad \text{and} \quad V(Y, Z) = 0. \]

Because (1.12) we have

\[ \text{These Riemann curvatures of } \overline{N} \text{ are thus always negative or zero. From now on we consider only the points } p \in N \text{ for which the total normal Riemann curvature of } \overline{N} \text{ is not zero (for the case } \mathcal{R} = 0, \text{ we can make an analogous remark as in 3a.). Since zero is an extremal value for the Riemann curvatures } K(X, \xi_p), \ X \in N_p \text{ and since the function } \delta: N_p \to N^\perp; \ X \to \overline{V}(X, \xi_p) \text{ is linear, we must suppose, if } \mathcal{R} \neq 0, \text{ that } m \geq 2n + 1, \text{ otherwise } \delta \text{ can not be injective. Consequently we have: if } m < 2n + 1, \text{ then } \mathcal{R} = 0 \text{ at each point of } N. \]

**Theorem 2.** Suppose that \( \overline{\omega} \) is a volume element of the spherical image of \( N \) at the point \( \eta(p) \) and that \( \omega \) is a volume element of \( N \) at the point \( p \), then

\[ (\eta^*(\overline{\omega}))^2 = (-1)^n \mathcal{R}(\omega)^2. \]

**Proof.** Since \( L = 0 \), we see, because of (2.1) and (2.2) that \( \eta_*(X) \) and \( \overline{V}(X, \xi_p), \ \forall X \in N_p \), have the same components with respect to the coordinate bases \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m} \) at the points \( \eta(p) \) and \( p \).
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Whereas \( \mathcal{K} \neq 0 \) at the point \( p \), \( \eta_* \) will be a bijection of \( N_p \) on \( S_{q(p)} \). Suppose that \( e_1, \cdots e_n \) is an orthonormal base field of \( N \), which determines the principal direction of the 2-covariant symmetric tensor field \( \langle Y, \bar{R}(Z, \xi) \bar{\xi} \rangle = -\langle \bar{V}(Y, \xi), \bar{V}(Z, \bar{\xi}) \rangle \). The dual forms of \( (e_i)_p, \cdots, (e_n)_p \) are denoted by \( \omega_1, \cdots, \omega_n \). Since

\[
[-\langle \bar{V}(e_i)_p, \bar{V}(e_j)_p \rangle, \bar{V}(e_i)_p, \bar{V}(e_j)_p] \]

becomes a diagonal matrix, we see that \( \eta_*((e_1)_p), \cdots, \eta_*((e_n)_p) \) are pairwise orthogonal. Consider the orthonormal base

\[
\eta_*((e_i)_p) / \langle \bar{V}(e_i)_p, \bar{V}(e_i)_p, \bar{V}(e_i)_p, \eta_* \rangle \]

and denote the dual base by \( \omega_1, \cdots, \omega_n \). Remark that \(-\langle \bar{V}(e_i)_p, \bar{V}(e_i)_p \rangle, \bar{V}(e_i)_p, \bar{V}(e_i)_p \rangle \) are the extremal values of the Riemann curvatures of \( \bar{N} \) at the point \( p \) in the two-dimensional directions of \( \bar{N}_p \) which contain \( \bar{\xi}_p \). We have

\[
\eta^*((\bar{\omega}_i)) = \langle \bar{V}(e_i)_p, \bar{V}(e_i)_p \rangle >^{1/2} \omega_i, \quad i = 1, \cdots, n
\]

and so we find

\[
\eta^*((\bar{\omega}_1 \cdots \bar{\omega}_n)) = \eta^*((\bar{\omega}_1)) \cdots \eta^*((\bar{\omega}_n)) = \prod_{i=1}^n \langle \bar{V}(e_i)_p, \bar{V}(e_i)_p \rangle >^{1/2} \omega_1 \cdots \omega_n = \sqrt{(-1)^n \mathcal{K}} \omega_1 \cdots \omega_n,
\]

and this completes the proof.

**Example.** A variable \( n \)-dimensional linear space \( N(s) \) which is dependent on one parameter \( s \), describes a monosystem \( \bar{N} \) in \( E^n \). If \( r(s) \) is a base curve and if \( a_1(s), \cdots, a_n(s) \) constitutes a base of the variable generating space \( N(s) \), then \( \bar{N} \) can (locally) be represented by

\[
X(s, 1_1, \cdots, 1_n) = r(s) + \sum_{i=1}^n 1_i a_i(s), \quad 1_i \in R, \quad i = 1, \cdots, n.
\]

Each generating space \( N(s) \) is a hypersurface of \( \bar{N} \), which is totally geodesic in \( E^n \). If (accents mean derivation to \( s \))

\[
\text{rank} \left[ r'(s) a_1(s) \cdots a_n(s) \right] = 2n + 1, \quad \forall s,
\]

then \( \bar{N} \) is non-developable. In this case it can be proved that at each point of each generating space \( N(s) \) we have \( \mathcal{K} \neq 0 \) and so we can apply Theorem 2 (see [4]).

**c.** \( N \) is totally geodesic in \( \bar{N} \) and \( \bar{V}(\xi_p, \xi_p) = 0 \), \( \forall p \in N \).

In this case the second fundamental form of \( N \) in \( \bar{N} \) is identically zero.
i.e. \( L = 0 \) at each point \( p \) of \( N \). If \( \nabla (\xi_p, \xi_p) = 0 \), \( \forall p \in N \), then the vector \( \xi_p \) determines at each point \( p \in N \) an asymptotic direction of \( N \). Because of (1.12), we find
\[
K(X, \xi_p) = -\frac{\langle \nabla (X, \xi_p), \nabla (X, \xi_p) \rangle}{\langle X, X \rangle}, \quad \forall X \in N_p
\]
These Riemann curvatures of \( N \) are always negative or zero. We consider only the points \( p \in N \), for which the total normal Riemann curvature \( \mathcal{K} \) of \( N \) is not zero and therefore we must suppose, analogously as in 3.b, that \( m \geq 2n + 1 \) (if \( m < 2n + 1 \), then we have again \( \mathcal{K} = 0 \) at each point of \( N \)).

**Theorem 3.** Suppose that \( \tilde{\omega} \) is a volume element of the spherical image \( S \) of \( N \) at the point \( \eta(p) \) and that \( \omega \) is a volume element of \( N \) at the point \( p \), then
\[
(\eta^*(\tilde{\omega}))^2 = (-1)^n \mathcal{K}(\omega)^2. \tag{3.2}
\]

**Proof.** The proof of this statement is totally analogous to that of Theorem 2.

**Example.** Suppose that \( x^1, \ldots, x^{2(n+1)} \) are orthonormal coordinates in \( E^{2(n+1)} \) and consider \( E^{n+1} \) as the subspace of \( E^{2(n+1)} \) represented by \( x^{n+2} = \ldots = x^{2(n+1)} = 0 \). In \( E^{n+1} \) we take a hypersurface \( N \), which is locally given by the following parametric representation (\( u_j \), \( j = 1, \ldots, n \) are the parameters)
\[
x^1 = f_i(u_1, \ldots, u_n), \quad i = 1, \ldots, n+1,
\]
\[
x^k = 0, \quad k = n+2, \ldots, 2(n+1). \tag{3.3}
\]
Using the unit normal vector field \( \tau(\tau_1(u_1, \ldots, u_n), \ldots, \tau_{n+1}(u_1, \ldots, u_n), 0, \ldots, 0) \) of \( N \) in \( E^{n+1} \), we construct the following \( (n+1) \)-dimensional submanifold \( N \) of \( E^{2(n+1)} \):
\[
x^1 = f_i(u_1, \ldots, u_n) \quad i = 1, \ldots, n+1,
\]
\[
x^j = 1 \tau_{j-n-1}(u_1, \ldots, u_n) \quad j = n+2, \ldots, 2(n+1) \text{ and } 1 \in R.
\]
Then \( N \) (or the part \( N \) given by (3.3), which will henceforth be denoted by \( N \)) clearly is a hypersurface of \( \bar{N} \).

Consider in \( E^{2(n+1)} \) the normal unit vector field \( \xi \) on \( N \), with components \( (0, \ldots, 0, \tau_1(u_1, \ldots, u_n), \ldots, \tau_{n+1}(u_1, \ldots, u_n)) \). If \( \bar{D} \) is the standard Riemann connection of \( E^{2(n+1)} \), \( \bar{D} \) the Riemann connection of \( \bar{N} \), \( \bar{\nabla} \) the second fundamental form of \( \bar{N} \), \( L \) the Weingarten map of \( N \) in \( \bar{N} \) and if \( \bar{L} \) is the Weingarten map of \( N \) in \( E^{n+1} \), then we have
\[
\bar{D} X \tau = \bar{L}(X), \quad \forall X \in N_p \tag{3.4}
\]
and the Gauss curvature \( G \) of \( N \) in \( E^{n+1} \) at the point \( p \) is given by \( \det \bar{L} \).
Moreover we find
\[ \overline{D}_X \xi - D_X \xi + \overline{V}(X, \xi) = L(X) + \overline{V}(X, \xi). \] (3.5)

But if we consider the components of the unit normal field \( \xi \), then it is clear that \( L = 0 \) at each point of \( N \), i.e., \( N \) is totally geodesic in \( \overline{N} \). We also have that \( \xi \) determines at each point of \( N \) an asymptotic direction of \( \overline{N} \), i.e., \( \overline{V}(\xi, \xi) = 0 \). It is also clear that
\[ \langle \overline{D}_X \tau, \overline{D}_X \tau \rangle = \langle D_X \xi, D_X \xi \rangle. \] (3.6)

Because of (3.4), (3.5) and (3.6) we find for the Riemann curvature of \( \overline{N} \) in the two-dimensional direction \((X, \xi_p)\) of \( \overline{N}_p \)
\[ \mathcal{K}(X, \xi_p) = -\langle \overline{V}(X, \xi_p), \overline{V}(X, \xi_p) \rangle \frac{\langle L(X), L(X) \rangle}{\langle X, X \rangle} = -\langle \overline{L}(X), \overline{L}(X) \rangle. \]

If the principal curvatures of the hypersurface \( N \) (of \( E^{n+1} \)) at the point \( p \) are denoted by \( 1/R_i \), \( i=1, \ldots, n \), then we have at once for total normal Riemann curvature of \( \overline{N} \) at \( p \)
\[ \mathcal{K} \equiv \sum_{i=1}^{n} (-1)^{n} \frac{1}{R_i^2} = (-1)^{n} G^2, \]
and this is what (3.2) says, because in our example \( \eta^*(\omega) = \pm |G| \omega \).

**Remark**

1. If \( n=1 \), then \( N \) is a curve on the surface \( \overline{N} \). Suppose that \( T \) is a unit tangent vector field of \( N \) and that the unit normal vector field \( \xi \) is parallel in the normal bundle \( N^1 \), then we have \( \overline{D}_T \xi = k T - L(T) \) for some \( k \in \mathbb{R} \) and Theorem 1 remains true \( (k = \det L \text{ and volume element is now arc element}) \). This is also valid for Theorems 2 and 3.

Remark that in the case \( n=1 \), the total normal Riemann curvature \( \mathcal{K} \) of \( \overline{N} \) at the point \( p \) of \( N \) is equal to the Riemann curvature (or Gauss curvature) \( G \) of \( \overline{N} \) at \( p \). We give an example for the third case (3.c): consider a non-developable ruled surface \( \overline{N} \) in \( E^n \) \( (n \geq 3) \), which is locally represented by
\[ r(s) + 1 \xi(s), \quad \xi^2 = 1, \quad s \in I \subset \mathbb{R}, \quad 1 \in \mathbb{R}, \]
where \( s \) is the arc length of the base curve \( I \rightarrow \overline{N}; s \rightarrow r(s) \), which is an orthogonal trajectory of the generating lines. Suppose that \( r(s) \) (which is in this example \( N \)) is a geodesic of \( \overline{N} \). Then, a theorem of Bonnet says that \( N \) is also the line of striction of \( \overline{N} \) and the conditions for all this are (with classical notations; accents mean derivation to \( s \)) \( r' \xi = r' \xi' = 0, \quad \forall s \in I \). In
this case the parameters of distribution \( d \) are given by \( d^2 = 1/\xi'^2 \), \( \forall \xi \in I \) and for the Riemann curvature \( K \) of \( \bar{N} \) at \( q \) we find \( G = -d^2/(d^2 + t^2)^2 \), where \( t \) is the distance between \( q \) and the point of striction on the generating line through \( q \). At the points \( p \) of \( N \) we have \( t=0 \) and thus

\[
G = -\frac{1}{d^2} = -\xi'^2 = -\frac{(d\xi)^2}{(d\xi)^2},
\]

and this is what (3.2) says.

2. Suppose that \( N \) is totally geodesic in \( E^m \) or that \( N \) is totally geodesic in \( \bar{N} \) and \( \bar{V}(\xi_q, \xi_q) = 0 \), \( \forall q \in N \). Take a point \( p \in N \) and a vector \( X \in T_pN \). Consider a curve \( \sigma : [a, +a] \rightarrow N; t \rightarrow \sigma(t) \) on \( N \), such that \( \sigma(0) = p \) and \( T\sigma(0) = X \). Then we have for the arc length \( s \) of \( \sigma \)

\[
\left( \frac{ds}{dt} \right)_{t=0} = <X, X>^{1/2}.
\]  

(3.7)

The spherical image of \( \sigma \) is the curve \( \eta \sigma \) on \( S \). We find for the arc length \( \bar{s} \) of the curve \( \eta \sigma \), because of (2.1) and (2.2),

\[
\left( \frac{d\bar{s}}{dt} \right)_{t=0} = <T_{\eta \sigma(0)}, T_{\eta \sigma(0)}>^{1/2}
\]

\[
= <\sum_{i=1}^{m} X(a^i) \left( \frac{\partial}{\partial x^i} \right)_{\eta(p)}, \sum_{i=1}^{m} X(a^i) \left( \frac{\partial}{\partial x^i} \right)_{\eta(p)}> 
\]

\[
= <\bar{V}(X, \xi_p), \bar{V}(X, \xi_p)>.
\]  

(3.8)

Now the expressions of \( \bar{K}(X, \xi_p) \) in the cases 3.b. and 3.c., together with (3.7) and (3.8) give a nice geometrical interpretation of such Riemann curvature of \( \bar{N} \): suppose that \( t=0 \) gives \( s=0 \)

\[
\bar{K}(X, \xi_p) = -\left( \frac{d\bar{s}}{ds} \right)_{s=0}^2.
\]

References

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