ON COMPLEX CONFORMAL CONNECTIONS IN AN ALMOST COMPLEX MANIFOLD WITH A TORSION TENSOR

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§ 1. Introduction

Let \( M \) be an \( n \)-dimensional Riemannian manifold with metric tensor \( g_{ji} \). The change of the metric

\[
\tilde{g}_{ji} = e^{\varphi} g_{ji},
\]

where \( \varphi \) is a certain scalar function, does not change the angle between two vectors at a point and so is called a conformal change of the metric.

If there exists a function \( \varphi \) such that the Riemannian manifold with metric tensor \( e^{2\varphi} g_{ji} \) is locally Euclidean, the Riemannian manifold is said to be conformally flat.

It is well known (Weyl [1]) that the so-called Weyl conformal curvature tensor

\[
W^h_{\kappa j i} = K^h_{\kappa j i} + \delta^h_k C_{ji} - \delta^h_j C_{ki} + C^h_k g_{ji} - C^h_j g_{ki}
\]

is invariant under a conformal change of \( g \), where \( K^h_{\kappa j i} \) is the Riemann–Christoffel curvature tensor of \( M \) and

\[
C_{ji} = -\left( \frac{1}{n-2} \right) K_{ji} + \left( \frac{1}{2(n-1)(n-2)} \right) K g_{ji},
\]

\[
C^{h}_{k} = C^{sh}_{k}, \quad K_{ji} = K_{ij}^t, \quad K = g^{ji} K_{ji}
\]

and a necessary and sufficient condition for \( M \) to be conformally flat is that

\[
W^h_{\kappa j i} = 0 \quad \text{for} \quad n > 3
\]

and

\[
\nabla_{j} C_{ji} - \nabla_{j} C_{ki} = 0 \quad \text{for} \quad n = 3,
\]

\( \nabla_{j} \) denoting the operator of covariant differentiation with respect to Christoffel symbols formed with \( g \).

A complex analogue of the above in a Kaehler manifold is given by K. Yano (K. Yano [2]).

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In a Kaehler manifold $M$ with the Hermitian metric tensor $g_{ij}$ and complex structure tensor $F_{ij}$, we have

$$V_k g_{ji} = 0, \ V_k F^k_{ij} = 0, \ V_k F_{ji} = 0,$$

where $F_{ij} = F_{ji} g_{st}$ and consequently $F_{ji} = -F_{ij}$.

The affine connection which satisfies

$$D_k e^s_{ji} = 0, \ D_k e^s_{F_{ji}} = 0$$

and torsion tensor

$$1/2 (F^h_{ji} - F^h_{ij}) = -E_{ji} q^h,$$

where $p$ is a scalar function and $q^h$ is a vector field, is given by

$$F^h_{ji} = \begin{bmatrix} h \\ j \\ i \\ \end{bmatrix} + p_j \partial_j^h + p_i \partial_i^h - p^h g_{ji} + q_j F^h_i + q_i F^h_j - q^h F_{ji},$$

where

$$p_i = \partial_i p, \ p^h = p_i g^h, \ q_i = -p_j F^i_j, \ q^h = q_i g^i.$$

For this connection called a complex conformal connection, K. Yano proved the following theorem:

*If in an $n$-dimensional Kaehler manifold $(n \geq 4)$, there exists a scalar function $p$ such that the complex conformal connection is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.*

Define $C^h_{kji}$ by

$$C^h_{kji} = K^h_{kji} + \delta^h_{k} L^i_{ji} - \delta^h_{j} L^k_{li} + q_j L^h_{ki} - q_i L^h_{kj} + F^h_k M_{ji} - F^h_j M_{ki} + M^h_{k} F_{ji} - M^h_{j} F_{ki}$$

$$+ \left( \frac{2}{n+4} \right) V_{kj} F^h_i + F_{kj} B^i_j,$$

where

$$V_{kj} = 1/2 K_{kj} F^i_i,$$

$$M_{ji} = -K_{ji} F^i_i,$$

$$M_{ji} = - \frac{1}{2 (n-2)} (H_{ji} + H_{ij}) - \frac{1}{n+4} V_{ji} + \frac{1}{n^2 - 4} \left( \frac{n+1}{n+4} V - 1/2 K \right) F_{ji},$$

$$L_{ji} = M_{ji} F^i_i,$$

$$L^h_k = L^h_{ki} g^{ih}, \ M^h_{k} = M_{kj} g^{ih}, \ B^h_i = B_{kj} g^{ih},$$

$$B_{ji} = H_{ji} + n M_{ji} - 2 M_{ij} - \frac{1}{n+2} \left( \frac{1}{n+4} V + 1/2 K \right) F_{ji},$$

$$V = V^u F^u \text{ and } K = K^u g^{u}.$$
This tensor is called a complex conformal curvature tensor. The complex conformal curvature tensor \( C^{kijh} \) is invariant under a change of the complex conformal connection. (O.K. Yoon [5]).

A complex analogue of the above in an almost complex manifold with a torsion tensor is not yet known. The main purpose of the present paper is to try to find some properties concerning this problem. In § 2 we state some of fundamental formulas in an almost complex manifold with a torsion tensor to fix our notation and in § 3, we introduce what we call complex conformal connections in an almost complex manifold with a torsion tensor.

In § 4, we study the integrability condition for an almost complex manifold \( M \) with a torsion tensor, in § 5, we state some fundamental formulas, and in § 6 we study an invariant curvature tensor.

**§ 2. Preliminaries**

We consider an \( n \)-dimensional almost complex manifold \( M \) with a torsion tensor \( S_{ji}^h = \frac{1}{2}(\Gamma_{ji}^h - \Gamma_{ij}^h) \) which is covered by a system of coordinate neighborhoods \( (U; \xi^h) \) and denote by \( g_{ji} \) and \( F_{ji}^h \) the components of the Hermitian metric tensor and those of the complex structure tensor of \( M \) respectively, where here and in the sequel the indices \( h, i, j, k, \ldots \) run over the range \( \{1, 2, \ldots, n\} \).

We denote by \( D_j \) the operator of covariant differentiation with respect to \( \Gamma_{ji}^h \), then the torsion tensor \( S_{ji}^h = \frac{1}{2}(\Gamma_{ji}^h - \Gamma_{ij}^h) \) is given by

\[
S_{ji}^h = -u^h F_{ji},
\]

where \( u^h \) are components of a vector field. Then, we have

\[
D_k g_{ji} = 0, \quad D_k F_{ji} = 0, \quad D_k F_{ji} = 0,
\]

where \( F_{ji} = F_{j'g_{si}} \) is skew symmetric.

Above all, we notice that an affine connection is symmetric, that is, which satisfies

\[
D_k g_{ji} = 0
\]

and whose torsion tensor

\[
1/2(\Gamma_{ji}^h - \Gamma_{ij}^h) = S_{ji}^h
\]

is uniquely determined and given by

\[
\Gamma_{ji}^h = \frac{1}{2} \left[ \left[ ji \right] + S_{ji}^h + S_{ji}^h + S_{ij}^h, \right.
\]

where (Hayden [6])

\[
S_{ji}^h = S_{ij}^k g_{ti}^k.
\]
So we have, for the components $\Gamma^h_{ji}$ of affine connection in an almost complex manifold $M$ with a torsion tensor $S_{ji}^h = -u^hF_{ji}$

\[(2.7) \quad \Gamma^h_{ji} = \{h\}_{ji} + u_jF^h_i + u_iF^h_j - u^hF_{ji},\]

where $\{h\}_{ji}$ are the Christoffel symbols formed with $g_{ji}$ and

\[F^h_j = g^h_iF_{ij} = -F_{jg}^h = -F^h_j, \quad u_i = u^i g_{ii}.\]

We denote by

\[(2.8) \quad K_{kji}^h = \partial_k \{h\}_{ji} - \partial_j \{h\}_{ki} + \{h\}_{s} \{s\}_{ji} - \{h\}_{s} \{s\}_{ki},\]

the components of the Riemann–Christoffel curvature tensor of $M$, where $\partial_k = \partial/\partial x^k$.

It is well known that $K_{kj}^i$ and $K_{kji}^h = K_{kji}^h g_{sh}$ satisfy

\[(2.9) \quad K_{kji}^h = -K_{kj}^i, \quad K_{kji}^h = -K_{kj}^i,\]

\[(2.10) \quad K_{kji}^h = K_{kji}^h,\]

\[(2.11) \quad K_{kji} + K_{jik} + K_{ijk} = 0\]

and

\[(2.12) \quad \nabla_v K_{kji}^h + \nabla_h K_{ji}^v + \nabla_j K_{ki}^h = 0,\]

\[(2.13) \quad \nabla_v K_{kji} = \nabla_v K_{ji} - \nabla_j K_{ki},\]

\[(2.14) \quad 2\nabla_v K_{ji} = \nabla_v K,\]

where

$K_{ji} = K_{ij} = K_{sji}^i$ and $K = g^{ji} K_{ji}$

are the Ricci tensor and the scalar curvature symbol $\{h\}_{ji}$, respectively.

§ 3. Integrability conditions

We consider the integrability conditions in an almost complex manifold $M$ with a torsion tensor.

From (2.7), using $D_k F_{ji} = 0$, we have

\[D_k F_{ji} = \nabla_k F_{ji} - u_i g_{kj} + u_j g_{ki} - w_j F_{ki} + w_i F_{kj} = 0,\]

where $w_j = -u_j F^s_j$, consequently

\[(3.1) \quad \nabla_k F_{ji} = g_{kj} u_i - g_{ki} u_j - F_{kj} w_i + F_{ki} w_j.\]
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Then, from the Ricci identity, we obtain

\begin{align*}
(3.2) \quad \nabla_i \nabla_j F_{ij} - \nabla_j \nabla_i F_{ij} &= K_{ik} F_{kj} - K_{k} F_{ij} \\
&= g_{kj} u_{li} - g_{lj} u_{ki} - g_{k} u_{ij} + g_{ij} u_{k} \\
&= F_{kj} w_{li} + F_{lj} w_{ki} + F_{k} w_{ij} - F_{ij} w_{k},
\end{align*}

where

\begin{align*}
(3.3) \quad u_{ji} &= \nabla_j u_{i} - u_{j} w_{i} + 1/2 \rho F_{ji}, \\
(3.4) \quad w_{ji} &= \nabla_j w_{i} - w_{j} w_{i} + 1/2 \rho g_{ji} = - u_{k} F_{ji}, \\
(3.5) \quad \rho = u_{k} g_{st} = w_{i} w_{s} g^{st}.
\end{align*}

If we define

\begin{align*}
(3.6) \quad K_{ij} F_{st} &= A_{ji}, \\
(3.7) \quad K_{j} F_{st} &= - H_{ji},
\end{align*}

then, from (2.9), (2.10) and (2.11), we have

\begin{align*}
(3.8) \quad K_{ij} F_{st} &= K_{ij} F_{st} = - 2 A_{ji}, \quad A_{ji} + A_{ij} = 0.
\end{align*}

And, from (3.2), we have also

\begin{align*}
(3.9) \quad A_{ji} - H_{ji} &= (n - 3) u_{ji} + u_{ij} + (u_{st} F_{st}) F_{ji},
\end{align*}

consequently

\begin{align*}
(3.10) \quad 2 A_{ji} - (H_{ji} - H_{ij}) &= (n - 4) (u_{ji} - u_{ij}) + 2 (u_{st} F_{st}) F_{ji}, \\
(3.11) \quad H_{ji} + H_{ij} &= - (n - 2) (u_{ji} + u_{ij}), \\
(3.12) \quad A - K &= 2 (n - 2) (u_{st} F_{st}),
\end{align*}

where \( A = u_{st} F_{st}, \quad K = K_{st} g^{st} = H_{st} F_{st}. \)

From (3.10), (3.11) and (3.12), we find

\begin{align*}
(3.13) \quad u_{ji} &= - \frac{1}{2(n-2)} (H_{ji} + H_{ij}) \\
&+ \frac{1}{2(n-4)} \left\{ 2 A_{ji} - (H_{ji} - H_{ij}) - \frac{A-K}{n-2} F_{ji} \right\} \quad \text{for } n > 4,
\end{align*}

\begin{align*}
(3.14) \quad 2 A_{ji} - (H_{ji} - H_{ij}) - \frac{A-K}{2} F_{ji} &= 0 \quad \text{for } n = 4.
\end{align*}

On the other hand, since the Nijenhuis tensor

\[ N_{ij} = F_j^t (\partial_i F^{\hat{s}}_t - \partial_i F^t_{\hat{s}}) - F^t_i (\partial_j F^{\hat{s}}_t - \partial_j F^t_{\hat{s}}) \]
of the complex structure tensor $F^h_i$ vanishes in virtue of (3.1), the complex structure tensor $F^h_i$ be integrable.

Thus, we have

**Proposition 3.1.** In an $n$-dimensional Riemannian manifold $M(n \geq 4$, even number) with metric tensor $g_{ji}$, the necessary and sufficient condition such that the manifold $M$ can be admissible an almost complex structure $F^h_i$ which satisfies

$$D_k g_{ji} = 0, \quad D_k F^h_i = 0 \quad \text{and} \quad \Gamma^h_\ell_j - \Gamma^h_\ell_i = -2F^h ji$$

is follows;

$$K^h_{kjhl} = -K^h_{jkh}, \quad K^h_{kjh} = -K^h_{kjh}, \quad K^h_{kjh} = K^h_{jhk},$$

$$K^h_{kjh} + K^h_{jkh} + K^h_{jk} = 0,$$

and

$$K^h_{kjh} F^h_i - K^h_{khi} K^h_j = g_{kj} u_{hi} - g_{jh} u_{ki} + g_{hi} u_{kj} - F^h_{kj} w_{hi} + F^h_{ih} w_{kj} - F^h_{ji} w_{h},$$

where, $u_{ji} = \frac{1}{2(n-2)} (H_{ji} + H_{ij})$,

$$w_{ji} = u_{ji} F^h_i.$$

If $n=4$, the last condition can be replaced by

$$2A_{ji} - (H_{ji} - H_{ij}) - \frac{A - K}{2} F^h_{ji} = 0,$$

where

$$A_{ji} = K^h_{khi} F^h_i, \quad K^h_{ji} = K^h_{khi}, \quad H_{ji} = -K^h_{jih};$$

and

$$A = A^h_{st} F^h_{st}, \quad K = K^h_{st} g^h_{st}.$$

§ 4. Some formulas in an almost complex manifold $M$ with a torsion tensor

We denote by

(4.1) \[ R^h_{kji} = \partial_k \Gamma^h_\ell_j - \partial_\ell \Gamma^h_\ell_i + \Gamma^h_\ell_i \Gamma^h_\ell_j - \Gamma^h_\ell_j \Gamma^h_\ell_i; \]

the components of the curvature tensor of $M$.

By a straightforward computation, from (2.7), we find
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(4.2) \[ R_{kji} = K_{kji} - F_k^h u_{ji} + F_j^h u_{ki} - u_k^h F_{ji} + u_j^h F_{ki} + F_i^h (u_k - u_{jk} - u_{kj} - u_{ki}) - 2 F_k^j (u_i w^h - u_i w_j) + u_k (\delta_j^h u_i - g_j^h u^h + F_j^h w_i - F_j^h w_i + F_i^h w_j) - u_j (\delta_i^h u_i - g_i^h u^h + F_i^h w_i - F_i^h w_i + F_i^h w_k), \]

where \( w^h = w_{i} g^h \), from which \( R_{kji} = R_{kji}^s g_{sh} \) satisfy

(4.3) \[ R_{kji} = -R_{jki}, \quad R_{kji} = -R_{kji}, \]

(4.4)

(4.5) \[ R_{kji} + R_{jki} + R_{ikj} = -2 F_{kj} X_{(ij)} + 2 F_{ji} X_{(kh)} + 2 F_{ki} X_{(jh)} - 2 F_{kj} X_{(ih)} + 2 F_{ji} X_{(kh)} - 2 F_{ki} X_{(jh)}, \]

where \( X_{(ji)} = X_{ji} + X_{ij}, \quad X_{[ji]} = X_{ji} - X_{ij}, \)

and

(4.6) \[ X_{ji} = u_{ji} + 2 u_{ij} w_i - u_{ij} w_j + (\delta/2) F_{ji}. \]

Using \( D_k F_{ji} = 0 \), from the Ricci identity, we have

(4.7) \[ R_{kji} F_{k} = R_{kji} F_{ji}. \]

Now, if we define

(4.8) \[ R_{tji} F_{ti} = -2 E_{ji}, \quad R_{jist} F_{st} = -2 V_{ji}, \quad R_{kji} F_{ki} = A_{ji}, \]

then, from (4.4) and (4.5), we have

(4.9) \[ 2 V_{ji} - 2 E_{ji} = -n X_{ji} + 2 X F_{ji}, \]

(4.10) \[ A_{ji} - A_{ij} = 2 V_{ji} = 2 X_{ji} - 2 X F_{ji}, \]

(4.11) \[ A_{ji} - E_{ji} = -(n - 2) X_{ji}, \]

where \( X = X_{ji} F_{ji} \), and consequently if we put

\[ A = A_{ji} F_{ji}, \quad E = E_{ji} F_{ji} = V_{ji} F_{ji}, \]

then we have

(4.12) \[ X = \frac{1}{n-2} (E - A). \]

From (4.11), we have

(4.13) \[ X_{[ji]} = \frac{1}{n-2} \{ 2 E_{ji} - (A_{ji} - A_{ij}) \}, \quad X_{(ji)} = -\frac{1}{n-2} (A_{ji} + A_{ij}). \]
furthermore, eliminating $X_{ji}$ from (4.9), (4.10) and (4.11) we have

$$n(A_{ji} - A_{ij}) - 4E_{ji} - 2(n - 2) V_{ji} + 2(E - A) F_{ji} = 0.$$  

Substituting (4.11) and (4.13) into (4.4) and (4.4), we have

$$R_{kji} = -R_{kij} = \frac{1}{n - 2} \left\{ F_{kj} A_{(ji)} - F_{jk} A_{(ki)} + F_{ji} A_{(kh)} - F_{ki} A_{(jh)} - F_{ih} (A_{kj} - 2E_{kj}) + F_{kj} (A_{ih} - 2E_{ih}) \right\},$$

$$R_{kj} + R_{jik} + R_{ikj} = \frac{2}{n - 2} \left\{ F_{kj} (A_{ih} - E_{ih}) + F_{ji} (A_{kh} - E_{kh}) + F_{ik} (A_{jh} - E_{jh}) \right\}.$$

On the other hand, from the Bianchi identity, we have

$$D_i R_{kji} + D_k R_{jih} + D_j R_{ikh} = 2u^s (F_{ls} R_{sjih} + F_{kj} R_{slih} + F_{jl} R_{skih}),$$

hence, by contracting with $F_{kj}$, we obtain

$$u^s R_{slih} = \frac{1}{n - 2} (F_{ust} D_{slih} - D_{ti} E_{ih}).$$

Substituting (4.18) into (4.17), we have

$$D_i R_{kji} + D_k R_{jih} + D_j R_{ikh} = \frac{2}{n - 2} \left\{ F_{ls} (F_{ust} D_{slih} - D_{ti} E_{ih}) + F_{kj} (F_{ust} D_{slih} - D_{ti} E_{ih}) + F_{jl} (F_{ust} D_{tlih} - D_{ks} E_{ih}) \right\}.$$

Therefore, we have the following

**Proposition 4.1.** In an $n$-dimensional ($n \geq 4$) almost complex manifold $M$ with Hermitian metric tensor $g_{ji}$, complex structure tensor $F_i^h$, the affine connection which satisfies

$$D_k g_{ji} = 0, \quad D_k F_{ji} = 0$$

and

$$\Gamma^h_{ji} - \Gamma^k_{ij} = -2F_{ji} u^h,$$

where $u^h$ is a vector field, is given by

$$\Gamma^h_{ji} = \left\{ \begin{array}{l} h \\ j \end{array} \right\} + F^h_{jti} + F^h_{ij} - F_{ji} u^h.$$
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Furthermore, for the curvature tensor, the following relations hold;
\[ R_{kjih} = - R_{jikh}, \quad R_{kjih} = - R_{kjh}, \quad R_{kji} F^i_h = R_{kjh} F^i_h, \]
\[ R_{kji} - R_{kjh} = \frac{1}{n-2} \{ F_{kh} A_{(ji)} - F_{jh} A_{(ki)} + F_{ji} A_{(kh)} - F_{ki} A_{(jh)} \}
- F_{ih} (A_{kj} - 2E_{kj}) + F_{kj} (A_{ih} - 2E_{ih}) \}
\]
\[ R_{kjih} + R_{jikh} + R_{ikjh} = \frac{2}{n-2} \{ F_{kj} (A_{ih} - E_{ih}) + F_{ji} (A_{kh} - E_{kh}) + F_{ik} (A_{jh} - E_{jh}) \}, \]
\[ nA_{ji} - 4E_{ji} - 2(n-2) V_{ji} + 2(E - A) F_{ji} = 0 \]

and
\[ D_l R_{kjih} + D_l R_{jikh} + D_l R_{lkh} = \frac{2}{n-2} \left( F_{lk} D_s R_{sjih} + F_{kj} D_s R_{ltih} + F_{ji} D_s R_{tkih} \right) F^{st} \]
\[ \quad - \frac{2}{n-2} \left( F_{lk} D_s E_{ih} + F_{kj} D_s E_{ih} + F_{ji} D_s E_{kh} \right). \]

§ 5. Complex conformal connections

In an almost complex manifold \( M \) with a torsion tensor \( S_{jih} = - u^h F_{ji} \), we consider a conformal change of Hermitian metric
\[ (5.1) \quad \tilde{g}_{ji} = e^{2p} g_{ji}, \quad \tilde{F}_{ji} = e^{2p} F_{ji}, \quad \tilde{F}_i^h = \tilde{F}_{js} g^{sh} = F_i^h, \]
where \( p \) is a scalar function, and we look for an affine connection \( \tilde{F}_{ji}^h \) such that
\[ (5.2) \quad \tilde{D}_k \tilde{g}_{ji} = 0, \quad \tilde{D}_k \tilde{F}_{ji} = 0, \]
where \( \tilde{D}_k \) are the operator of covariant differentiation with respect to the connection \( \tilde{F}_{ji}^h \), and the torsion tensor \( \tilde{S}_{ji}^h \) is given by
\[ (5.3) \quad \tilde{S}_{ji}^h = - v^h \tilde{F}_{ji}, \]
where \( v^h \) are components of a vector field. We call such a metric change a complex conformal change of the metric.

By the remark above, we have, for the components \( \tilde{F}_{ji}^h \) of this affine connection,
\[ (5.4) \quad \tilde{F}_{ji}^h = \begin{bmatrix} h \\ j_{ji} \end{bmatrix} + \tilde{F}_{j_{ji}}^h v_i + \tilde{F}_{ji}^h v_j - v^h \tilde{F}_{ji}, \]
where \( \begin{bmatrix} h \\ j_{ji} \end{bmatrix} \) are the Christoffel symbols with \( \tilde{g}_{ji} \), and \( v_i = v^s \tilde{g}_{si} \).
Also, using
\begin{equation}
\tilde{h}_{ji}^{\tilde{h}} = \tilde{h}_{ji} + p_j \tilde{e}_j^h + p_i \tilde{e}_i^h - p^h g_{ji},
\end{equation}
we have
\begin{equation}
\tilde{F}_{ji}^{\tilde{h}} = \tilde{h}_{ji} + p_j \tilde{e}_j^h + p_i \tilde{e}_i^h - p^h g_{ji} + v_j \tilde{F}_i^{\tilde{h}} + v_i \tilde{F}_j^{\tilde{h}} + v^h \tilde{F}_{ji},
\end{equation}
where \( p_i = \partial_i p \) and \( p^h = p e^{ih} \). From (2.7), using (5.1), we have
\begin{equation}
\tilde{F}_{ji}^{\tilde{h}} = \Gamma_{ji}^{\tilde{h}} + p_j \tilde{e}_j^h + p_i \tilde{e}_i^h - p^h g_{ji} + \left( (v_j - u_j) F_i^{\tilde{h}} + (v_i - u_i) F_j^{\tilde{h}} - (e^{2p} t - u^h) F_{ji} \right).
\end{equation}
If we define \( q^h \) by
\begin{equation}
q^h = e^{2p} t - u^h,
\end{equation}
then
\begin{equation}
q_i = q^h g_{si} = (e^{2p} t - u^h) g_{si} = e^{2p} g_{si} - u^h g_{si} = v_i - u_i.
\end{equation}
Substituting (5.8) and (5.9) into (5.7), we have
\begin{equation}
\tilde{F}_{ji}^{\tilde{h}} = \Gamma_{ji}^{\tilde{h}} + p_j \tilde{e}_j^h + p_i \tilde{e}_i^h - p^h g_{ji} + q_j F_i^{\tilde{h}} + q_i F_j^{\tilde{h}} - q^h F_{ji}.
\end{equation}
We now compute \( D_k \tilde{F}_{ji} \) and find
\begin{align*}
D_k \tilde{F}_{ji} &= D_k (e^{2p} F_{ji}) \\
&= e^{2p} \left( (q_i F_j^t - p_j) F_{ki} + (p_i - q_i F^t) F_{kj} \\
&\quad - (q_i + p_i F^t) g_{kj} + (q_j + p_j F^t) g_{kj} \right).
\end{align*}
Thus, in order that \( D_k \tilde{F}_{ji} = 0 \), we must have
\begin{equation}
(q_i F_j^t - p_j) F_{ki} + (p_i - q_i F^t) F_{kj} - (q_i + p_i F^t) g_{kj} + (q_j + p_j F^t) g_{ki} = 0
\end{equation}
for which, transvecting with \( g^{kj} \), we find
\begin{equation}
(n-2) (p_i F^t + q_i) = 0,
\end{equation}
that is, assuming \( n \geq 4 \)
\begin{equation}
p_i F^t + q_i = 0.
\end{equation}
Therefore,
\begin{equation}
q_i = -p_i F^t, \quad p_i = q_i F^t.
\end{equation}
The converse being evident, we have

\begin{proposition}
In an almost complex manifold with a torsion tensor...
\end{proposition}
by a complex conformal change of the metric
\[ \tilde{g}_{ij} = e^{2p} g_{ij}, \quad \tilde{F}_{ij} = e^{2p} F_{ij}, \]
the affine connection \( \tilde{\Gamma}_j{}^i{}^h \) which satisfies
\[ \bar{D}_k \tilde{g}_{ij} = 0, \quad \bar{D}_k \tilde{F}_{ij} = 0 \]
and
\[ \tilde{\Gamma}_j{}^i{}^h - \tilde{\Gamma}_i{}^j{}^h = -2v^h \tilde{F}_{ij}, \]
where \( p \) is a scalar function and \( v^h \) is a vector field, is given by
\[ \tilde{\Gamma}_j{}^i{}^h = \Gamma_j{}^i{}^h + p_i \delta_j{}^h + p_j \delta_i{}^h - p^h g_{ji} + q_j F_i{}^h + q_i F_j{}^h - q^h F_{ij} \]
where
\[ p_i = \partial_i p, \quad p^h = p_i \delta^i{}^h, \quad q_i = -p_i F_i{}^t, \quad q^h = q_i \delta^i{}^h. \]

We call such an affine connection a complex conformal connection in an almost complex manifold with a torsion tensor.

§ 6. Curvature tensor of a complex conformal connection and its invariant

We consider a complex conformal connection (5.10) in an almost complex manifold with a torsion tensor and compute the curvature tensor of \( \tilde{\Gamma}_j{}^i{}^h \):
\[
\tilde{R}^h{}_{jki} = \partial_k \tilde{\Gamma}^h{}_{j}{}^{i}{}^{k} - \partial_j \tilde{\Gamma}^h{}_{k}{}^{i}{}^{k} + \tilde{\Gamma}^h{}_{k}{}^{i}{}^{l} \tilde{\Gamma}^l{}_{j}{}^{k}{}^{i} - \tilde{\Gamma}^h{}_{j}{}^{i}{}^{l} \tilde{\Gamma}^l{}_{k}{}^{i}{}^{k}.
\]

By a straightforward computation, we find
\[
\tilde{R}^h{}_{jki} = R^h{}_{jki} + \partial_j p^h \delta^i{}^k - \partial_k p^h \delta^j{}^i - q_j p_k + q_k p_j + q^h \delta^i{}^k + F_j{}^k q_i - F_i{}^k q_j - F_i{}^q q_j + F_k{}^q q_i + F_i{}^h \alpha_j{}^k - 2 F_{kj} \delta_i{}^h,
\]
where
\[
\begin{align*}
\rho_i &= D_j p_i - p_j p_i + q_j q_i + (\lambda/2) g_{ji}, \quad \rho_i = p_i, \\
\lambda &= p_i p^i = q_i q^i, \\
q_i &= -p_i F_i{}^t, \quad \rho_i = q_i F_i{}^t, \\
\alpha_{kj} &= q_{kj} - q_{jk} - \mu F_{kj}, \quad \alpha_{kj} = -\alpha_{kj}, \\
\mu &= \lambda + 2 q_i u^i, \\
\beta_{ji} &= p_j q_i - p_i q_j + q_j w_i - q_i w_j + p_j u_i - u_j p_i.
\end{align*}
\]
If we define

\[ \beta_j^h = \beta_j^h, \beta_j^i = -\beta_j^i. \]

If we define

\[ \tilde{R}_{kji}^j = \tilde{R}_{kji}^j \]

\[ R_{stji}^u = -2E_{ji}, \tilde{R}_{stji}^u = -2\tilde{V}_{ji}, \tilde{R}_{stji}^u = \tilde{A}_{ji}, \]

then, from (4.8), we have

\[ \tilde{E}_{ji} = E_{ji} - 2(q_{ji} - q_{ij}) - (\alpha/2) F_{ji} + n\beta_{ji}, \]

where \( \alpha = \alpha_{st} F_{st} = 2p - n\mu, p = \rho_{st} e^{t} = q_{st} F_{st} \), and transvecting with \( F_{ji} (= e^{2p} F_{ji}) \), we have

\[ e^{2p} E = (2\beta - \alpha), \]

where \( \alpha = \alpha_{st} F_{st} = 2p - n\mu, \beta = \beta_{st} F_{st} = 2\mu \) and

\[ (6.13) \quad e^{2p} E = E - (n + 4) p + \frac{n(n + 4)}{2} \mu. \]

From (4.8), (6.11) and (6.2), we have

\[ \tilde{V}_{ji} = V_{ji} - 2(q_{ji} - q_{ij}) + 2\mu F_{ji} - (n/2) \alpha_{ji}, \]

\[ \tilde{A}_{ji} = A_{ji} - (n - 1)q_{ji} + q_{ij} - pF_{ji} - \alpha_{ji} + 2\beta_{ji} \]

and transvecting with \( F_{ji} (= e^{2p} F_{ji}) \), we have

\[ e^{2p} V = V - (n + 4) p + \frac{n(n + 4)}{2} \mu, \]

\[ e^{2p} A = A - 2(n + 1) p + (n + 4) \mu. \]

From (6.16) and (6.17), we find

\[ (6.18) \quad p = \frac{1}{2(n^2 - 4)}(nA - 2E) - \frac{1}{2(n^2 - 4)}(nA - 2E)e^{2p}, \]

\[ (6.19) \quad \mu = \frac{1}{(n + 4)(n^2 - 4)} \{ (n + 4) A - 2(n + 1) E \} \]

\[ - \frac{1}{(n + 4)(n^2 - 4)} \{ (n + 4) A - 2(n + 1) E \} e^{2p}. \]

If we put

\[ Q = \frac{1}{2(n^2 - 4)}(nA - 2E), \quad \bar{Q} = \frac{1}{2(n^2 - 4)}(nA - 2E), \]

\[ N = \frac{(n + 4) A - 2(n + 1) E}{(n + 4)(n^2 - 4)}, \quad \bar{N} = \frac{(n + 4) A - 2(n + 1) E}{(n + 4)(n^2 - 4)}, \]
On complex conformal connections in an almost complex manifold with a torsion tensor

(6.18) and (6.19) can be written as

\[ p = Q - Qe^{2\theta}, \quad \mu = N - \overline{N}e^{2\theta} \]

and consequently

\[ PF_{ji} = QF_{ji} - \overline{Q}F_{ji}, \quad \mu F_{ji} = NF_{ji} - \overline{N}F_{ji}. \]

From (6.15), we have

\[ \overline{A}_{ji} = \overline{A}_{ij} = A_{ji} - A_{ij} - 2(n+2)(q_{ji} - q_{ij}) - 2P - 2\mu F_{ji} + 4\beta_{ji}, \]

from which, using (6.12) and (6.21), we have

\[ q_{ji} - q_{ij} = \frac{1}{(n+4)(n-2)} \{ n(A_{ji} - A_{ij}) - 4E_{ji} - 2(n-2)\overline{Q}F_{ji} \} \]

\[ - \frac{1}{(n+4)(n-2)} \{ n(\overline{A}_{ji} - \overline{A}_{ij}) - 4E_{ji} - 2(n-2)\overline{Q}F_{ji} \}, \]

\[ \beta_{ji} = \frac{1}{2(n+4)(n-2)} \left[ 4(A_{ji} - A_{ij}) - 2(n+2)F_{ji} \right. \]

\[ + (n-2) \left[ 2\overline{Q} - (n+4)N \right] F_{ji} \left. \right] - \frac{1}{2(n+4)(n-2)} \left[ 4(\overline{A}_{ji} - \overline{A}_{ij}) \right. \]

\[ - 2(n+2)E_{ji} + (n-2) \left[ 2\overline{Q} - (n+4)\overline{N} \right] F_{ji} \]

On the other hand, from (6.15), we have

\[ \overline{A}_{ji} + \overline{A}_{ij} = A_{ji} + A_{ij} - (n-2)(q_{ji} + q_{ij}), \]

from which, using (6.23), we find

\[ q_{ji} = \frac{1}{2(n+4)(n-2)} \left\{ (n+4)(A_{ji} + A_{ij}) + n(A_{ji} - A_{ij}) \right. \]

\[ - 4E_{ji} - 2(n-2)QF_{ji} \right\} \]

\[ - \frac{1}{2(n+2)(n-2)} \left\{ (n+4)(\overline{A}_{ji} + \overline{A}_{ij}) + n(\overline{A}_{ji} - \overline{A}_{ij}) - 4\overline{E}_{ji} - 2(n-2)\overline{Q}F_{ji} \right\}. \]

Substituting (6.23) into (6.6), we obtain

\[ \alpha_{ji} = \frac{1}{(n+4)(n-2)} \left[ n(A_{ji} - A_{ij}) - 4E_{ji} - (n-2) \left[ 2Q + (n+4)N \right] F_{ji} \right] \]

\[ - \frac{1}{(n+4)(n-2)} \left[ n(\overline{A}_{ji} - \overline{A}_{ij}) - 4\overline{E}_{ji} - (n-2) \left[ 2\overline{Q} + (n+4)\overline{N} \right] F_{ji} \right]. \]

If we define $M_{ji}$, $L_{ji}$, $B_{ji}$ and $T_{ji}$ by

...
\[ M_{ji} = \frac{1}{2(n+4)(n-2)} \left\{ (n+4)(A_{ji} + A_{ij}) + n(A_{ji} - A_{ij}) ight\} - 4E_{ji} - 2(n-2)QF_{ji} \], (6.28)

\[ L_{ji} = M_{ji}F'_{ji} \], (6.29)

\[ B_{ji} = \frac{1}{2(n+4)(n-2)} \left\{ 4(A_{ji} - A_{ij}) - 2(n+2)E_{ji} + (n-2)2Q - (n+4)N \right\} F_{ji} \], (6.30)

\[ T_{ji} = \frac{1}{2(n+4)(n-2)} \left\{ n(A_{ji} - A_{ij}) - 4E_{ji} - (n-2)2Q + (n+4)N \right\} F_{ji} \], (6.30)

then, (6.24), (6.26) and (6.27) can be written as

\[ p_{ji} = B_{ji} - \overline{B}_{ji}, \quad q_{ji} = M_{ji} - \overline{M}_{ji}, \quad \alpha_{ji} = T_{ji} - \overline{T}_{ji} \], (6.32)

and consequently

\[ p_{ji} = q_{ji}F_{ji}, \quad q_{ji} = (M_{ji} - \overline{M}_{ji})F_{ji} = L_{ji} - \overline{L}_{ji} \], (6.33)

\[ L_{jh} = L_{js}F_{sh}, \quad M_{jh} = M_{js}F_{sh}, \quad B_{jh} = B_{js}F_{sh} \], (6.34)

Substituting (6.32), (6.33) and (6.34) into (6.2), we have

\[ R_{kh} + \delta_{j}^{h}L_{ki} - \delta_{k}^{h}L_{ji} - \overline{\delta}_{j}^{h}\overline{L}_{ki} + \overline{\delta}_{k}^{h}\overline{L}_{ji} + \overline{\delta}_{j}^{h}\overline{L}_{ki} + \overline{\delta}_{k}^{h}\overline{L}_{ji} \]
\[ \quad + F_{j}^{h}M_{ki} - F_{k}^{h}M_{ji} + F_{ji}M_{k}^{h} + F_{kj}M_{i}^{h} \]
\[ \quad + F_{j}^{h}T_{kj} - 2F_{kj}B_{i}^{h} \]

\[ = R_{kj} + \delta_{j}^{h}L_{ki} - \delta_{k}^{h}L_{ji} - g_{ji}L_{k}^{h} + g_{ki}L_{j}^{h} \]
\[ \quad + F_{j}^{h}M_{ki} - F_{k}^{h}M_{ji} + F_{ji}M_{k}^{h} + F_{ki}M_{j}^{h} \]
\[ \quad + F_{j}^{h}T_{kj} - 2F_{kj}B_{i}^{h} \]

If we define \( C_{j}^{h} \) as

\[ C_{j}^{h} = R_{kj} + \delta_{j}^{h}L_{ki} - \delta_{k}^{h}L_{ji} - g_{ji}L_{k}^{h} + g_{ki}L_{j}^{h} \]
\[ \quad + F_{j}^{h}M_{ki} - F_{k}^{h}M_{ji} + F_{ji}M_{k}^{h} + F_{ki}M_{j}^{h} \]
\[ \quad + F_{j}^{h}T_{kj} - 2F_{kj}B_{i}^{h} \], (6.35)

then (6.35) reduces into

\[ C_{j}^{h} = C_{j}^{k} + \delta_{j}^{h}L_{ki} - \delta_{k}^{h}L_{ji} - g_{ji}L_{k}^{h} + g_{ki}L_{j}^{h} \]
\[ \quad + F_{j}^{h}M_{ki} - F_{k}^{h}M_{ji} + F_{ji}M_{k}^{h} + F_{ki}M_{j}^{h} \]
\[ \quad + F_{j}^{h}T_{kj} - 2F_{kj}B_{i}^{h} \]

We call such a tensor \( C_{j}^{h} \) a complex conformal curvature tensor in an
almost complex manifold with a torsion tensor. Thus, we have the following theorem.

**Theorem** In an $n$-dimensional ($n \geq 4$) almost complex manifold with a torsion tensor, the complex conformal curvature tensor $C_{kji}^h$ defined by

$$C_{kji}^h = R_{kji}^h + \delta_j^k L_{ki} - \delta_k^h L_{ji} - \epsilon_{ji} L_{k}^h + \epsilon_{ki} L_j^h$$

$$+ F_j^k M_{ki} - F_h^k M_{ji} - F_{ji} M_{k}^h + F_{ki} M_j^h$$

$$+ F_i^h T_{kj} - 2F_{ij} B_i^h$$

is an invariant under a complex conformal change of metric

$$\bar{g}_{ji} = e^{2\varphi} g_{ji}, \quad \bar{F}_{ji} = e^{2\varphi} F_{ji}.$$

### Reference


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