ON THE DEFINITION OF A HYPERATOM OF A RING

By Alexander Abian

In the direct product decomposition of a (not necessarily associative or commutative) ring \( R \) essential use is made of the notion of a hyperatom \([1],[3],[4]\) where a hyperatom of \( R \) is defined by the conjunction of statements (1) and (2) below. We show here that in all the cases pertaining to \([1],[3],[4]\), statement (1) implies statement (2). Accordingly, we define a hyperatom subject to statement (1) alone.

REMARK. We call a (not necessarily associative or commutative) ring zero-product-associative if and only if a product of elements of the ring which is equal to zero remains equal to zero no matter how its factors are associated. In \([2]\) it is shown that if \( A \) is a zero-product-associative ring without nilpotent elements then a product of elements of \( A \) which is equal to zero remains equal to zero no matter how its factors are associated or permuted. We observe also \([4, \text{Lemma} \ 2]\) that an alternative ring without nilpotent elements is zero-product-associative.

DEFINITION. A nonzero element \( a \) of a (not necessarily associative or commutative) ring \( A \) is called a hyperatom of \( A \) if and only if for every element \( x \) of \( A \),

\[ (1) \quad ax \neq 0 \text{ implies } a(xs) = a \text{ for some } s \in A \]

THEOREM. Let \( A \) be a zero-product-associative ring without nilpotent elements and let \( a \) be a hyperatom of \( A \). Then for every nonzero element \( x \) of \( A \),

\[ (2) \quad ax = x^2 \text{ implies } a = x \]

PROOF. Since \( x \neq 0 \) and \( A \) has no nilpotent elements, \( ax = x^2 \) implies \( ax \neq 0 \), and since \( a \) is a hyperatom \( a(xs) = a \) for some \( s \in A \) by (1). From \( ax = x^2 \) it follows \((a-x)x = 0\) which, by the Remark, implies \((a-x)xas = (a-x)(a(xs)) = 0\) and therefore \((a-x)a = 0\). But then the latter together with \((a-x)x = 0\) imply
\( (a-x)^2 = 0 \). Hence \( a = x \), as desired.

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REFERENCES


