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INFINITESIMAL CL-VARIATIONS OF SASAKIAN HYPERSURFACES OF A KAEHLERIAN MANIFOLD

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0. Introduction

In recent years a number of authors has studied infinitesimal variations of submanifolds of Riemannian and Kaehlerian manifolds (cf. [1], [2], [3]).

On the other hand, Y. Tashiro and S. Tachibana showed some characteristic properties of Fubinian and C-Fubinian manifolds in their paper [6], where the notion of C-loxodromes was introduced in an almost contact manifold with affine connection.

The purpose of the present paper is to study infinitesimal variations of a Sasakian hypersurface of a Kaehlerian manifold which carry C-loxodromes to C-loxodromes. Such an infinitesimal variation will be called in this paper an infinitesimal CL-variation.

In §1, we state some properties on almost contact metric structures induced on a hypersurface from the Kaehlerian structure of the ambient manifold.

In §2, we consider infinitesimal variations of a hypersurface of a Kaehlerian manifold and obtain variations of structure tensors of almost contact metric structure induced on the hypersurface, and those of the Christoffel symbols and second fundamental form of the hypersurface.

In the last §3, we consider an infinitesimal CL-variation which carries a Sasakian hypersurface of a Kaehlerian manifold into a hypersurface of the same kind.

1. Preliminaries

$\overline{M}^{2n+2}$ $(n>1)$ be a real $(2n+2)$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and $F_i^{\ h}$ be its almost complex structure tensor and g_{ji} its almost Hermitian metric tensor, where here and in the sequel, the indices $h, i, j, k, \ldots$ run over the range $\{\bar{1}, \bar{2}, \ldots, \overline{2n+2}\}$. Then we have

$$(1.1) \qquad F_i^{\ t} F_t^{\ h} = -\delta_i^h, \quad F_j^{\ t} F_i^{\ s} g_{ts} = g_{ji},$$

(1.2) $$\nabla_j F_i{}^h=0,$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols Γ^h_{ji} formed with g_{ji}.

Let M^{2n+1} be a $(2n+1)$-dimensional orientable Riemannian manifold covered by a system of coordinate neighborhoods $\{V\ ;y^a\}$ and g_{cb} be its fundamental metric tensor, where here and in the sequel, the indices $a,b,c,\ldots.$ run over the range $\{1,2,\ldots,2n+1\}$. We assume that M^{2n+1} is isometrically immersed into $\overline{M}^{2n+2}$ by the immersion $i:M^{2n+1}\longrightarrow\overline{M}^{2n+2}$ and identify $i(M^{2n+1})$ with M^{2n+1}. We represent the immersion by

$$x^h=x^h(y^a)$$

and put

$$B_b{}^h=\partial_b x^h,\quad \partial_b=\partial/\partial y^b$$

Then $B_b{}^h$ are $2n+1$ linearly independent vectors of $\overline{M}^{2n+2}$ tangent to M^{2n+1}. Since the immersion i is isometric, we have

(1.3) $$g_{cb}=g_{ji}B_c{}^jB_b{}^i.$$

We represent the unit normal to M^{2n+1} by C^h. We then have

(1.4) $$g_{ji}B_b{}^j C^i=0,\ g_{ji}C^jC^i=1.$$

As to the transforms of $B_b{}^h$ and C^h by $F_i{}^h$ we have

(1.5) $$F_i{}^hB_b{}^i=f_b{}^a B_a{}^h+f_bC^h,\ F_i{}^hC^i=-f^aB_a{}^h,$$

where f_b^a is a tensor field of type (1,1), f_b a 1-form of M^{2n+1} and $f^a=f_bg^{ba}$ a vector field of M^{2n+1}, g^{ba} being contravariant components of g_{cb}.

From (1.3), (1.4) and (1.5) we get

(1.6) $$f_a{}^b=B_a{}^iF_i{}^hB_b{}^h,\ f_b=B_b{}^iF_i{}^hC_h,$$

where $B_a{}^h=B_b{}^ig^{ba}g_{ih}$, $C_h=g_{ih}C^i$.

Denoting by Γ^a_{cb} the Christoffel symbols formed with g_{cb}, then it is well known that Γ^h_{ji} and Γ^a_{cb} are related by

(1.7) $$\Gamma^a_{cb}=(\partial_cB_b{}^h+\Gamma^h_{ji}B^{ji}_{cb})B_a{}^h$$

where $B^{ji}_{cb}=B_c{}^jB_b{}^i$.

We define the van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ and C^h along M^{2n+1} by

(1.8) $$\nabla_cB_b{}^h=\partial_cB_b{}^h+\Gamma^h_{ji}B^{ji}_{cb}-\Gamma^a_{cb}B_a{}^h,\nabla_cC^h=\partial_cC^h+\Gamma^h_{ji}B_c{}^jC^i,$$

respectively.

Then the equations of Gauss and Weingarten are written as

(1.9) $$\nabla_c B_b{}^h = h_{cb} C^h, \quad \nabla_c C^h = -h_c{}^a B_a{}^h,$$

where h_{cb} is the second fundamental tensor of M^{2n+1} and $h_c{}^a = h_{cb} g^{ba}$.

Applying F to the both sides of (1.5) and using (1.1), (1.3), (1.4) and (1.5), we find

(1.10) $$\begin{cases} f_b{}^e f_e{}^a = -\delta_b^a + f_b f^a, \ f_b{}^e f_e = 0, \ f_e{}^a f^e = 0, \ f_e f^e = 1, \\ f_c{}^e f_b{}^d g_{ed} = g_{cb} - f_c f_b. \end{cases}$$

Equation (1.10) shows that the set $(f_b{}^a,\ g_{cb},\ f_b)$ defines the so-called almost contact metric structure on M^{2n+1}.

Differentiating (1.6) covariantly along M^{2n+1} and taking account of (1.9), we find out

(1.11) $$\nabla_c f_b{}^a = -h_{cb} f^a + h_c{}^a f_b, \quad \nabla_c f_b = -h_{ce} f_b^e.$$

we now consider the tensor field $S_{cb}{}^a$ defined by

(1.12) $$S_{cb}{}^a = N_{cb}{}^a + (\nabla_c f_b - \nabla_b f_c) f^a,$$

where $N_{cb}{}^a$ is the Nijenhuis tensor formed with f_b^a:

$$N_{cb}{}^a = f_c^e \nabla_e f_b{}^a - f_b^e \nabla_e f_c{}^a - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^a.$$

When the tensor field S_{cb}^a vanishes identically, the almost contact metric structure $(f_b^a,\ g_{cb},\ f_b)$ is said to be *normal*.

Thus if the almost contact metric structure is normal, then we have from (1.10), (1.11) and (1.12)

(1.13) $$f_c{}^e h_e{}^a - h_c^e f_e{}^a = 0, \quad h_e{}^a f^e = h f^a,$$

where $h = h_{cb} f^c f^b$.

When the almost contact metric structure $(f_b{}^a,\ g_{cb},\ f_b)$ satisfies

(1.14) $$\nabla_c f_b - \nabla_b f_c = 2 f_{cb},$$

the structure is said to be *contact*.

Substituting (1.11) into (1.14), We obtain

(1.15) $$f_c{}^e h_e{}^a + h_c{}^e f_e{}^a = 2 f_c^a.$$

When the almost contact metric structure is normal and contact, we call the structure a *Sasakian structure.* For a Sasakian structure, we have (1.13), (1.15) and consequently

$$f_c{}^e h_e{}^a = f_c{}^a,$$

from which, transvecting $f_b{}^c$,

$$h_{cb} = g_{cb} + (h-1) f_c f_b. \tag{1.16}$$

Thus (1.11) becomes

$$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c^a f_b, \quad \nabla_c f_b = f_{cb}. \tag{1.17}$$

Consequently by using the Ricci identity, we have

$$f_a K_{dcb}{}^a = f_d g_{cb} - f_c g_{db}, \quad f_a K_c{}^a = 2n f_c, \tag{1.18}$$

$K_{dcb}{}^a$ and K_{ba} being the curvature and Ricci tensor of M^{2n+1} respectively.

On the other hand, equations of Gauss and Codazzi are respectively written as

$$K_{dcb}{}^a = K_{kji}{}^h B_{dcbh}^{kjia} + h_d{}^a h_{cb} - h_c{}^a h_{db}, \tag{1.19}$$

$$K_{kji}{}^h B_{dcb}^{kji} C_h = \nabla_d h_{cb} - \nabla_c h_{db},$$

$K_{kji}{}^h$ being the curvature tensor of $\overline{M}^{2n+2}$, where $B_{dcbh}^{kjia} = B_d{}^k B_c{}^j B_b{}^i B_a{}^h$, $B_{dcb}^{kji} = B_d{}^k B_c^j B_b{}^i$.

2. Infinitesimal variations of hypersurface of a Kaehlerian manifold

Now we consider an infinitesimal variation of the hypersurface M^{2n+1} in $\overline{M}^{2n+2}$ given by

$$\bar{x}^h = x^h + \xi^h(y)\varepsilon, \tag{2.1}$$

ξ^h being a vector field of $\overline{M}^{2n+2}$ defined along M^{2n+1}, where ε is an infinitesimal, We then have

$$\overline{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\overline{B}_b{}^h = \partial_b \bar{x}^h$ are $2n+1$ linearly independent vectors tangent to the varied hypersurface at the varied point $(\bar{x}^h)$.

We displae vectors $\overline{B}_b{}^h$ parallely from the varied point $(\bar{x}^h)$ to the original point (x^h) and obtain the vectors

$$\widetilde{B}_b{}^h = \overline{B}_b{}^h + \Gamma_{ji}^h(x+\xi\varepsilon)\xi^j \overline{B}_b{}^i \varepsilon$$

at the point (x^h), or

(2.2) $$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon,$$

neglecting the terms of order higher than one with respect to ε, where

(2.3) $$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}^h B_b{}^j \xi^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε.

Putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h.$$

we have from (2.2)

(2.4) $$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon.$$

If we put

$$\xi^h = \xi^a B_a{}^h + \lambda C^h,$$

we have

(2.5) $$\nabla_b \xi^h = (\nabla_b \xi^a - \lambda h_b{}^a) B_a{}^h + (\nabla_b \lambda + h_{be}\xi^e) C^h,$$

where ξ^a and λ being respectively a vector field and a scalar function on M^{2n+1}.

When the tangent space at a point (x^h) of the submanifold and that at the corresponding point $(\bar{x}^h)$ of the deformed submanifold are parallel, that is, $\nabla_b \lambda + h_{be}\xi^e = 0$, we say that the variation is *parallel*. And if $\xi^a = 0$, that is, if the variation vector ξ^h is normal to the submanifold, we say that the variation is *normal* ([1]).

Now we denote by $\bar{C}^h$ the unit normal to the varied hypersurface and by $\tilde{C}^h$ the vector obtained from $\bar{C}^h$ by parallel displacement of $\bar{C}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

(2.6) $$\tilde{C}^h = \bar{C}^h + \Gamma_{ji}^h (x + \xi\varepsilon) \xi^j \bar{C}^i \varepsilon.$$

We put

$$\delta C^h = \tilde{C}^h - C^h.$$

Then δC^h, being orthogonal to C^h, is of the form

$$\delta C^h = \eta^a B_a{}^h \varepsilon,$$

η^a being a vector field on M^{2n+1}. Thus from (2.6) and the last two equations, we have

(2.7) $$\bar{C}^h = C^h - \Gamma^h_{ji}\xi^j C^i\varepsilon + \eta^a B_a{}^h\varepsilon.$$

Applying the operator δ to $B_b{}^j C^i g_{ji}=0$ and using (2.4), (2.5) and $\delta g_{ji}=0$, we find

$$\eta_b = -(\nabla_b\lambda + h_{ba}\xi^a),$$

where $\eta_b=\eta^c g_{cb}$. Thus we have

(2.8) $$\delta C^h = -(\nabla^a\lambda + h_b{}^a\xi^b)B_a{}^h\varepsilon,$$

(2.9) $$\bar{C}^h = C^h - \Gamma^h_{ji}\xi^j C^i\varepsilon - (\nabla^a\lambda + h_b{}^a\xi^b)B_a{}^h\varepsilon,$$

where $\nabla^a = g^{ab}\nabla_b$.

Now applying the operator δ to (1.5) and using $\delta F_i{}^h=0$, (1.5), (1.11), (2.4), (2.5) and (2.8), we have

(2.10) $$\begin{cases} \delta f_b{}^a = [\mathcal{L} f_b{}^a + \lambda(f_b{}^e h_e{}^a - h_b{}^e f_e^a) + f_b(\nabla^a\lambda) - (\nabla_b\lambda)f^a]\varepsilon, \\ \delta f_b = [\mathcal{L} f_b - \lambda h_b{}^e f_e - f_b{}^e\nabla_e\lambda]\varepsilon, \\ \delta f^a = [\mathcal{L} f^a + (\nabla^e\lambda)f_e{}^a + \lambda f^e h_e{}^a]\varepsilon, \end{cases}$$

where $\mathcal{L} f_b^a$, $\mathcal{L} f_b$, $\mathcal{L} f^a$ denote respectively the Lie derivatives of f_b^a, f_b and f^a with respect to the vector field ξ^a in M^{2n+1} ([1], [2]), those are

(2.11) $$\begin{cases} \mathcal{L} f_b^a = \xi^e\nabla_e f_b{}^a + (\nabla_b\xi^e)f_e{}^a - (\nabla_e\xi^a)f_b{}^e, \\ \mathcal{L} f_b = \xi^e\nabla_e f_b + (\nabla_b\xi^e)f_e, \\ \mathcal{L} f^a = \xi^e\nabla_e f^a - (\nabla_e\xi^a)f^e. \end{cases}$$

A variation of a submanifold for which $\delta f_b^a=0$ is said to be *f-preserving* ([2]).

Applying the operator δ to (1.3) and using $\delta g_{ji}=0$, (2.4) and (2.5), we get

(2.12) $$\delta g_{cb} = [\mathcal{L} g_{cb} - 2\lambda h_{cb}]\varepsilon,$$

where $\mathcal{L} g_{cb}$ denotes the Lie derivative of g_{cb} with respect to ξ^a, that is,

(2.13) $$\mathcal{L} g_{cb} = \nabla_c\xi_b + \nabla_b\xi_c, \quad (\xi_b = \xi^c g_{cb}).$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric* and that for which δg_{cb} is proportional to g_{cb} is said to be *conformal* ([1]).

On the other hand, It is well known that ([1], [2]) the variations of the

Christoffel symbols and the second fundamental form of the hypersurface M^{2n+1} are of the form

$$(2.14)\qquad \begin{cases} \delta\Gamma^a_{cb}=[\pounds\Gamma^a_{cb}-\nabla_c(\lambda h_b{}^a)-\nabla_b(\lambda h_c{}^a)+\nabla^a(\lambda h_{cb})]\varepsilon, \\ \delta h_{cb}=[\pounds h_{cb}+\nabla_c\nabla_b\lambda+\lambda(K_{kji}{}^hC^kB^{ji}_{cb}C_h-h_{ce}h_b{}^e)]\varepsilon, \end{cases}$$

where $\pounds\Gamma^a_{cb}$ and $\pounds h_{cb}$ denote respectively the Lie derivatives of Γ^a_{cb} and h_{cb}, these are

$$(2.15)\qquad \begin{cases} \pounds\Gamma^a_{cb}=\nabla_c\nabla_b\xi^a+K_{dcb}{}^a\xi^d, \\ \pounds h_{cb}=\xi^d\nabla_dh_{cb}+h_{ce}\nabla_b\xi^e+h_{eb}\nabla_c\xi^e. \end{cases}$$

A variation of a submanifold for which $\delta\Gamma^a_{cb}=0$ is said to be *affine* ([1]).

3. Infinitesimal normal CL-variations of Sasakian hypersurfaces of a Kaehlerian manifold

Let $x^h=x^h(y^a(t))$ be a curve on M^{2n+1}, parametrized by any parameter t. Then we have

$$(3.1)\qquad \frac{dx^h}{dt}=B_a{}^h\frac{dy^a}{dt},$$

$$(3.2)\qquad \frac{d^2x^h}{dt^2}=(\partial_cB_b{}^h)\frac{dy^c}{dt}\frac{dy^b}{dt}+B_b{}^h\frac{d^2y^b}{dt^2}.$$

Substituting (1.8) and (1.9) into (3.2) we have

$$\frac{d^2x^h}{dt^2}+\Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}=h_{cb}C^h\frac{dy^c}{dt}\frac{dy^b}{dt}+\Big(\frac{d^2y^a}{dt^2}+\Gamma^a_{cb}\frac{dy^c}{dt}\frac{dy^b}{dt}\Big)B_a{}^h.$$

Thus the C-loxodrome in an almost contact metric space M^{2n+1} in terms of parameter t is represented by the form in $\overline{M}^{2n+2}$

$$\frac{d^2x^h}{dt^2}+\Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}=h_{cb}C^h\frac{dy^c}{dt}\frac{dy^b}{dt}+\Big(\alpha(t)\frac{dy^a}{dt}+bf_cf^a_b\frac{dy^c}{dt}\frac{dy^b}{dt}\Big)B_a{}^h,$$

where $\alpha(t)$ is a function of t and b is a constant ([4], [6]).

Now we write it down in the form

$$(3.3)\qquad \frac{d^2x^h}{dt^2}+\Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}=h_{cb}C^h\frac{dy^c}{dt}\frac{dy^b}{dt} +\Big\{\alpha(t)\frac{dy^a}{dt}+A(f_cf^a_b+f_bf^a_c)\frac{dy^c}{dt}\frac{dy^b}{dt}\Big\}B_a{}^h.$$

If the variation carries C-loxodromes in M^{2n+1} into C-loxodromes in the deformed manifold, we can find

$$\frac{d^2\bar{x}^h}{dt^2}+\Gamma^h_{ji}(\bar{x})\frac{d\bar{x}^j}{dt}\frac{d\bar{x}^i}{dt}=\bar{h}_{cb}\bar{C}^h\frac{dy^c}{dt}\frac{dy^b}{dt} \tag{3.4}$$
$$+\left\{\beta(t)\frac{dy^a}{dt}+B(\bar{f}_c\bar{f}_b{}^a+\bar{f}_b\bar{f}_c{}^a)\frac{dy^c}{dt}\frac{dy^b}{dt}\right\}\bar{B}_a{}^h,$$

where $\beta(t)$ is a function of t and B is a constant.

Since $\frac{d\bar{x}^h}{dt}=\frac{dx^h}{dt}+(\partial_c\xi^h)\frac{dy^c}{dt}\varepsilon$, we obtain

$$\frac{d^2\bar{x}^h}{dt^2}=\frac{d^2x^h}{dt^2}+(\partial_c\partial_b\xi^h)\frac{dy^c}{dt}\frac{dy^b}{dt}\varepsilon+(\partial_b\xi^h)\frac{d^2y^b}{dt^2}\varepsilon. \tag{3.5}$$

Substituting (3.5) into (3.4), we have by the straightforward computation,

$$\bar{\Gamma}^a_{cb}-\Gamma^a_{cb}=\{\beta(t)-\alpha(t)\}\frac{dy^a}{dt} \tag{3.6}$$
$$+B(\bar{f}_c f_b{}^a+\bar{f}_b\bar{f}_c{}^a)\frac{dy^c}{dt}\frac{dy^b}{dt}-A(f_cf_b^a+f_bf_c^a)\frac{dy^c}{dt}\frac{dy^b}{dt}.$$

Denoting by $\delta\Gamma^a_{cb}=\bar{\Gamma}^a_{cb}-\Gamma^a_{cb}$, $\delta f_b^a=\bar{f}_b^a-f_b^a$ and $\delta f_c=\bar{f}_c-f_c$, and using the usuall progress for the equation derived from (3.6), we can obtain

$$\delta\Gamma^a_{cb}=\rho_c\delta^a_b+\rho_b\delta^a_c+\alpha(f_cf_b^a+f_bf_c^a)+\beta\{\delta(f_cf_b^a+f_bf_c^a)\}, \tag{3.7}$$

where α and β are constants, and ρ_c is a vector field.

Such a variation will be called an *infinitesimal CL-variation* in this paper.

Contracting a and c in (3.7) and using (2.14), we see that

$$2(n+1)\rho_c=\nabla_c(\nabla_e\xi^e-\lambda h_e{}^e)\varepsilon,$$

which denotes that ρ_c is a gradient.

From (3.7) we have

$$\nabla_d\delta\Gamma^a_{cb}=(\nabla_d\rho_c)\delta^a_b+(\nabla_d\rho_b)\delta^a_c+\alpha(f_c^a\nabla_df_b+f_b\nabla_df_c^a+f_b^a\nabla_df_c+f_c\nabla_df_b^a)$$
$$+\beta\nabla_d[\delta(f_cf_b^a+f_bf_c^a)].$$

Substituting this into the following identity [3]

$$\delta K_{dcb}{}^a=\nabla_d\delta\Gamma^a_{cb}-\nabla_c\delta\Gamma^a_{db}, \tag{3.8}$$

and using the structure equations we get

$$\delta K_{dcb}{}^a=\delta^a_c\nabla_d\rho_b-\delta^a_d\nabla_c\rho_b+\alpha(2f_b^af_{dc}+f_c{}^af_{db}-f_d{}^af_{cb}) \tag{3.9}$$

$$+\alpha(2f_c f_b \delta_d^a - f_c f^a g_{db} - 2f_d f_b \delta_c^a + f_d f^a g_{cb})$$
$$+\beta\nabla_d\{\delta(f_c f_b^a + f_b f_c^a)\} - \beta\nabla_c\{\delta(f_d f_b^a + f_b f_d^a)\},$$

and contracting d and a in (3.9) we have

$$\delta K_{cb} = -2n\nabla_c\rho_b - 2\alpha\{g_{cb} - (2n+1)f_c f_b\} + \beta\nabla_d\{\delta(f_c f_b^d + f_b f_c^d)\}. \tag{3.10}$$

Transvecting (3.9) with f_a, we obtain

$$f_a \delta K_{dcb}{}^a = f_c\nabla_d\rho_b - f_d\nabla_c\rho_b + \alpha(f_d g_{cb} - f_c g_{db}) \tag{3.11}$$
$$+\beta f_a\nabla_d\{\delta(f_c f_b^a + f_b f_c^a)\} - \beta f_a\nabla_c\{\delta(f_d f_b^a + f_b f_d^a)\}.$$

Now we assume that the infinitesimal CL-variation carries a Sasakian hypersurface into a hypersurface of the same kind. Taking the operator δ of the both sides of the first equations of (1.18) and substituting (3.11) into the equations thus obtained, we get

$$K_{dcb}{}^a(\delta f_a) = (\delta f_d)g_{cb} + f_d\delta g_{cb} - (\delta f_c)g_{db} - f_c\delta g_{db} \tag{3.12}$$
$$-f_c\nabla_d\rho_b + f_d\nabla_c\rho_b - \alpha(f_d g_{cb} - f_c g_{db})$$
$$-\beta f_a\nabla_d\{\delta(f_c f_b^a + f_b f_c^a)\} + \beta f_a\nabla_c\{\delta(f_d f_b^a + f_b f_d^a)\}.$$

Transvecting this with g^{cb}, we then have

$$K_d{}^a(\delta f_a) = 2n(\delta f_d) + f_d(g^{cb}\delta g_{cb} + \nabla_a\rho^a) \tag{3.13}$$
$$-f^a(\delta g_{da} + \nabla_a\rho_d) - 2n\alpha f_d + \beta(f_c g_{db} - f_d g_{cb})(\delta g^{cb}).$$

In the same way, transvecting (3.12) with f^d and using (1.18), we get

$$\delta g_{cb} = -\nabla_c\rho_b + f_c(f^a\delta g_{ab} + f^a\nabla_a\rho_b) + \alpha(g_{cb} - f_c f_b) \tag{3.14}$$
$$+\beta f^d f_a\nabla_d\{\delta(f_c f_b{}^a + f_b f_c^a)\} - \beta f^d f_a\nabla_c\{\delta(f_d f_b^a + f_b f_d^a)\}.$$

Taking the skew-symmetric part of (3.14) and transvecting the equations thus obtained with f^a and substituting this into (3.14), we have

THEOREM 3.1. *In a Sasakian hypersurfaces of a Kaehlerian manifold with an infinitesimal CL-variation which carries a Sasakian hypersurface into a hypersurface of the same kind, the following relation holds*:

$$\delta g_{cb} = -\nabla_c\rho_b + \gamma f_c f_b + \alpha(g_{cb} - f_c f_b) \tag{3.15}$$
$$+\beta f^d f_a\nabla_d\{\delta(f_c f_b^a + f_b f_c^a)\} - \beta f^d f_a\nabla_c\{\delta(f_d f_b^a + f_b f_d^a)\},$$

where $\gamma=f^a f^b(\delta g_{ab}+\nabla_a\rho_b)$.

THEOREM 3.2. *If the infinitesimal CL-variation carrying a Sasakian hypersurface into a hypersurface of the same kind is parallel and normal, the following relation holds:*

$$\delta g_{cb}=-\nabla_c\rho_b+\theta(g_{cb}+f_c f_b) \tag{3.16}$$

where $\theta=\alpha-\lambda h\beta\varepsilon=constant.$

PROOF. If the variation is parallel and normal, (3.7) becomes

$$\delta\Gamma^a_{cb}=\rho_c\delta^a_b+\rho_b\delta^a_c+\theta(f_c f^a_b+f_b f^a_c) \tag{3.17}$$

because of (1.10) and (2.10), where $\theta=\alpha-\lambda h\beta\varepsilon$.

From (3.17) we have

$$\begin{aligned}\delta K_{dcb}{}^a=&\delta^a_c\nabla_d\rho_b-\delta^a_d\nabla_c\rho_b+\theta(2f^a_b f_{dc}+f^a_c f_{db}-f^a_d f_{cb})\\&+\theta(2f_c f_b\delta^a_d-f_c f^a g_{db}-2f_d f_b\delta^a_c+f_d f^a g_{cb})\\&+(f_c f^a_b+f_b f^a_c)(\nabla_d\theta)-(f_d f^a_b+f_b f^a_d)(\nabla_c\theta),\end{aligned} \tag{3.18}$$

$$\delta K_{cb}=-2n\nabla_c\rho_b-2\theta\{g_{cb}-(2n+1)f_c f_b\}+(f_c f^a_b+f_b f^a_c)(\nabla_a\theta). \tag{3.19}$$

In the same way, we have

$$\delta g_{cb}=-\nabla_c\rho_b+f_c(f^a\delta g_{ab}+f^a\nabla_a\rho_b)+\theta(g_{cb}-f_c f_b), \tag{3.20}$$

from which

$$f^b(\delta g_{cb}+\nabla_c\rho_b)=\gamma f_c.$$

Substituting this into (3.20) we have

$$\delta g_{cb}=-\nabla_c\rho_b+\gamma f_c f_b+\theta(g_{cb}-f_c f_b). \tag{3.21}$$

From the last equations of (1.18), taking the operator δ of the variation, we have

$$K_c{}^d(\delta f_d)+f_d(\delta K_c{}^d)=2n(\delta f_c).$$

This equation can be represented as following:

$$f_d K_{ce}(\delta g^{de})+f^d(\delta K_{cd})=2n(\delta f_c)-K_c{}^d(\delta f_d). \tag{3.22}$$

Substiting (3.19) and (3.21) into (3.22) and using (1.10) and (1.18), we obtain

$$f_d K_{ce}\nabla^d\rho^e-2n\gamma f_c-2nf^d\nabla_c\rho_d+4n\theta f_c+f^d_c\nabla_d\theta=0.$$

Transvecting this with f^c and using (1.18), we get $2\theta=\gamma$.

Substituting this results into the identity ([3])

$$\nabla_d\delta g_{cb}=g_{ce}\delta\Gamma^e_{db}+g_{be}\delta\Gamma^e_{dc}$$

and using (1.17), we get

$$-\nabla_d\nabla_c\rho_b+(g_{cb}+f_cf_b)(\nabla_d\theta)=2\rho_dg_{cb}+\rho_cg_{db}+\rho_bg_{dc}. \tag{3.23}$$

By the Ricci identity, this equation is written as

$$K_{dcba}\rho^a+(g_{cb}+f_cf_b)(\nabla_d\theta)-(g_{db}+f_df_b)(\nabla_c\theta)=\rho_dg_{cb}-\rho_cg_{db}, \tag{3.24}$$

where $\rho^a=g^{ab}\rho_b$.

Transvecting this with f^cf^b, and using the last equations of (1.18), we have

$$\nabla_d\theta=(f^e\nabla_e\theta)f_d. \tag{3.25}$$

And again transvecting (3.24) with g^{db}, we can have

$$K_c{}^e\rho_e=2n(\rho_c-\nabla_c\theta),$$

by the virtue of (3.25). Transvecting this with f^c and using (1.18), we get

$$f^e\nabla_e\theta=0. \tag{3.26}$$

Thus (3.25) and (3.26) complete our theorem.

Now we are going to prove our main theorm.

THEOREM 3.3. *Let M^{2n+1} be a Sasakian hypersurface of a Kaehlerian manifold $\overline{M}^{2n+2}$. If an infinitesimal parallel and normal CL-variation, or an f-preserving infinitesimal CL-variation for which $\delta f_c=0$ carries a Sasakian hypersurface M^{3n+1} into a hypersurface of the same kind, then the following five statements are equivalent:*

(1) *the variation is isometric,*

(2) *the variation is affine,*

(3) *the variation is conformal,*

(4) *the variation preserves the curvature tensor,*

(5) *the variation preserves the Ricci tensor.*

PROOF. It is sufficient to show that at the case the infinitesimal CL-variation is parallel and normal. In fact, if the variation is f-preserving and $\delta f_c=0$, (3.7) and (3.17) show $\theta=\alpha$.

If the variation is affine, then we see from (3.17) by contracting a and c

$$\rho_c=0 \text{ and } \theta=0.$$

Consequently from (3.16) we obtain that the variation is isometric.

Now we are going to prove that if the variation is confromal then it is isometric since the isometric variation is confromal ([1]).

By this assumption and (3.16) we get

$$\nabla_c\rho_b=(\theta-2A\varepsilon)g_{cb}+\theta f_c f_b, \tag{3.27}$$

where A is a scalar.

From (3.27) we have

$$f^b\nabla_c\rho_b=Bf_c, \quad B=2(\theta-A\varepsilon). \tag{3.28}$$

Differentiating this covariantly, we can have

$$f_d{}^b(\nabla_c\rho_b)+f^b(\nabla_d\nabla_c\rho_b)=(\nabla_d B)f_c+Bf_{dc}.$$

Substituting (3.23) into this and transvecting $f^d f_a{}^c$, we have

$$\rho_e f_a{}^e=0,$$

from which

$$\rho_a=(\rho_e f^e)f_a.$$

Again differentiating this covariantly and using the structure equations and (3.28), we have

$$\nabla_c\rho_a=Bf_a f_c+(\rho_e f^e)f_{ca}.$$

Since ρ_a is gradient, the last equation shows that

$$\rho_e f^e=0,$$

which implies $\rho_a=0$, and consequently we have from (3.28), $\theta=A\varepsilon$.

Using this results we get from (3.27)

$$\theta(g_{cb}-f_c f_b)=0,$$

which implies $\theta=0$ and consequently implies that the variation is isometric. Thus these results and (3.18) show that the variation preserves the curvature and Ricci tensor.

Finally we assume that the variation preserves the Ricci tensor, then (3.19) can be represented as following:

$$n\nabla_c\rho_b=-\theta\{g_{cb}-(2n+1)f_c f_b\},$$

from which, we see that

(3.29) $$f^b\nabla_c\rho_b=2\theta f_c.$$

Comparing (3.28) with (3.29), we can easily find out

$$\rho_b=0,\quad \theta=0.$$

These complete our theorem.

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