ANTI-HOLOMORPHIC SUBMANIFOLDS OF A SASAKIAN MANIFOLD WITH VANISHING C-BOCHNER CURVATURE TENSOR.

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In [9], Yano proved

**THEOREM A.** Let $M^n$, $n \geq 5$, be an anti-invariant submanifold of a Sasakian manifold $M^{2n-1}$ with vanishing C-Bochner curvature tensor. If the second fundamental tensors of $M^n$ commute, then $M^n$ is locally a product of a conformally flat Riemannian space and a 1-dimensional space.

**THEOREM B.** Let $M^n$, $n \geq 4$, be a totally umbilical anti-invariant submanifold normal to the structure vector field $\xi$ of a Sasakian manifold $M^{2n+1}$ with vanishing C-Bochner curvature tensor. Then $M^n$ is conformally flat.

**THEOREM C.** Let $M^n$, $n \geq 4$, be an anti-invariant submanifold normal to the structure vector field $\xi$ of a Sasakian manifold $M^{2n+1}$ with vanishing C-Bochner curvature tensor. If the second fundamental tensors commute, then $M^n$ is conformally flat.

The purpose of the present paper is to prove the following Theorem 1, 2 and 3 corresponding to Theorems A, B and C by replacing the condition that the submanifold is anti-invariant with that is anti-holomorphic respectively.

**THEOREM 1.** Let $M^n$, $n \geq 5$, be an anti-holomorphic submanifold tangent to the structure vector field $\xi$ of a Sasakian manifold $M^{2n-1}$ with vanishing C-Bochner curvature tensor. If the second fundamental tensors of $M^n$ commute, then $M^n$ is locally a product of a conformally flat Riemannian space and a 1-dimensional space.

**THEOREM 2.** Let $M^n$, $n \geq 4$, be a totally umbilical anti-holomorphic submanifold of a Sasakian manifold $M^{2n+1}$ with vanishing C-Bochner curvature tensor. Then $M^n$ is conformally flat.

**THEOREM 3.** Let $M^n$, $n \geq 4$, be an anti-holomorphic submanifold of a Sasakian manifold $M^{2n+1}$ with vanishing C-Bochner curvature tensor. If the second fund-
amental tensors of \( M^n \) commute, then \( M^n \) is conformally flat.

1. C-Bochner curvature tensor

We first of all recall definition and fundamental properties of Sasakian manifolds for later use. Let \( M^{2m+1} \) be a \((2m+1)\)-dimensional differentiable manifold of class \( C^\infty \) covered by a system of coordinate neighborhoods \( \{ U : \gamma \} \) (the indices \( \alpha, \beta, ..., \kappa, \lambda, \mu, ... \) run over the range \( \{ 1, 2, ..., 2m+1 \} \)) in which there are given a tensor field \( \phi^\kappa_\lambda \) of type \((1,1)\), a vector field \( \xi^\kappa \), a 1-form \( \eta_\lambda \) and a Riemannian metric tensor \( g_{\mu\lambda} \) satisfying

\[
\begin{align*}
\phi^\kappa_\lambda \phi^\lambda_\mu &= -\delta^\kappa_\mu + \eta_\mu \xi^\kappa, \\
\phi^\kappa_\lambda \xi^\lambda &= 0, \quad \eta_\lambda \phi^\lambda_\mu = 0, \quad \eta_\lambda \xi^\lambda = 1, \\
g_{\gamma\delta} \phi^\gamma_\mu \phi^\delta_\lambda &= g_{\mu\lambda} - \eta_\mu \eta_\lambda, \\
\xi^\lambda &= g_{\lambda\kappa} \xi^\kappa.
\end{align*}
\]

If

\[
\begin{align*}
\nabla_\lambda \xi^\kappa &= \phi^\kappa_\lambda, \\
\nabla_\mu \phi^\kappa_\lambda &= -g_{\mu\lambda} \xi^\kappa + \delta^\kappa_\mu \xi^\lambda.
\end{align*}
\]

where \( \nabla_\lambda \) denotes the operator of covariant differentiation with respect to \( g_{\mu\lambda} \), then such a set \((\phi^\kappa_\lambda, \xi^\kappa, \eta_\lambda, g_{\mu\lambda})\) is called a normal contact structure. Such a manifold \( M^{2m+1} \) is called a Sasakian manifold. In view of the last equation of (1.1) we shall write \( \xi_\lambda \) instead of \( \eta_\lambda \) in the sequel. In a Sasakian manifold, the tensor field \( \phi^\kappa_\mu = \phi^\kappa_\mu \eta_\mu \) is skew-symmetric.

It is well known that in a Sasakian manifold equation (1.2) and the Ricci identity give

\[
\begin{align*}
K_{\mu\lambda \kappa} \xi^\lambda &= \delta^\kappa_\mu \xi_\nu - \delta^\kappa_\nu \xi^\mu, \\
K_{\mu \kappa} \xi^\lambda &= 2m \xi^\mu, \\
K_{\mu \alpha} \phi^\alpha_\lambda + K_{\kappa \alpha} \phi^\alpha_\mu &= 0,
\end{align*}
\]

where \( K_{\mu\kappa \lambda} \) and \( K_{\mu \lambda} \) are the curvature tensor and the Ricci tensor of the manifold respectively.

As an analogue of the Bochner curvature tensor in a Kaehlerian manifold, the C-Bochner curvature tensor in a Sasakian manifold is defined (cf. [9]) by

\[
\begin{align*}
B_{\mu\nu\lambda} \xi^\kappa &= K_{\mu\kappa \lambda} \xi^\lambda + (\phi^\alpha_\nu \xi^\kappa - \phi^\alpha_\nu \xi^\kappa) L_{\mu\lambda} - (\phi^\alpha_\nu \xi^\kappa - \phi^\alpha_\nu \xi^\kappa) L_{\nu\lambda} \\
&\quad + L_{\nu} (g_{\mu\lambda} - \xi^\mu \xi_\lambda) - L_{\mu} (g_{\nu\lambda} - \xi^\nu \xi_\lambda).
\end{align*}
\]
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\[ +\phi^k_\mu M^\mu_{\mu\lambda} - \phi^\mu_{\mu\lambda} M^\mu_k + M^\mu_\mu \phi_{\mu\lambda} - M^\mu_\mu \phi_{\mu\lambda} - 2(\phi_{\mu\mu} M^\mu_k + M^\mu_\mu \phi_{\lambda k}) + (\phi^k_\mu \phi_{\mu\lambda} - \phi^\mu_{\mu\lambda} \phi_{\mu\lambda} - 2\phi_{\mu\mu} \phi^k_{\lambda k}), \]

where

(1.7) \[ L_{\mu\lambda} = \frac{1}{2(m+2)} [-K_{\mu\lambda} - (L+3)g_{\mu\lambda} + (L-1)\xi_{\mu} \xi_{\lambda}], \quad L^\kappa = L_{\mu\alpha} g^\alpha_{\kappa}, \]

(1.8) \[ L = g^{\mu\lambda} L_{\mu\lambda}, \]

(1.9) \[ M^\mu_{\mu\lambda} = -L_{\mu\alpha} \phi^\alpha_{\lambda}, \quad M^\mu_k = M_{\mu\alpha} g^\alpha_{\kappa}. \]

From (1.7) and (1.8), we have

(1.10) \[ L = \frac{-K + 2(3m+2)}{4(m+1)}, \]

where \( K \) is the scalar curvature of the manifold.

Using (1.4), we have from (1.7)

(1.11) \[ L_{\mu\lambda} \xi^\lambda = -\xi^\mu. \]

From the first equation of (1.9) and (1.11), we have

(1.12) \[ M^\mu_{\mu\lambda} \phi^\alpha_{\lambda} = L_{\mu\alpha} \xi^\alpha_{\mu}. \]

It is easily verify that the C-Bochner curvature tensor satisfies the following identities:

\[ B^\mu_{\nu\lambda\kappa} = -B^\mu_{\nu\lambda\kappa}, \quad B^\mu_{\lambda\nu\kappa} + B^\mu_{\lambda\kappa\nu} + B^\kappa_{\lambda\nu\mu} = 0, \quad B_{\alpha\mu\lambda} = 0, \quad B_{\nu\mu\lambda\kappa} = -B_{\nu\mu\lambda\kappa}, \]

\[ B_{\nu\mu\lambda\kappa} = B_{\lambda\nu\mu} \xi^\kappa = 0, \quad B_{\nu\mu\alpha} \phi^\alpha_{\lambda} = B_{\nu\mu\lambda} \phi^\alpha_{\alpha}, \quad B_{\nu\mu\lambda} \phi^\mu_{\kappa} = 0, \]

where \( B_{\nu\mu\lambda\kappa} = B_{\nu\mu\lambda}^\alpha g^\alpha_{\kappa} \).

2. Anti-holomorphic submanifolds of a Sasakian manifold

We consider an \( n \)-dimensional Riemannian manifold \( M^n, \quad n \geq 1 \), covered by a system of coordinate neighborhoods \( \{V, y^h\} \) (the indices \( h, i, j, \ldots \) run over the range \( \{1, 2, \ldots, n\} \)) and isometrically immersed in a Sasakian manifold \( M^{2m+1} \) and denote the immersion by

(2.1) \[ x^k = x^k(y^h). \]

We put

(2.2) \[ B^i_t = \partial_i x^k (\partial_t = \partial/\partial y^i) \]

and denote by \( C_y 2m+1-n \) mutually orthogonal unit vectors normal to \( M^n \) (the
indices \( x, y, z \) run over the range \( \{(n+1), \ldots, (2m+1)\} \). Then the metric tensor \( g_{ji} \) of \( M^n \) and that of the normal bundle are respectively given by

\[
g_{ji} = g_{\mu\lambda} B_{ji}^{\mu\lambda}, \quad g_{zy} = g_{\lambda\mu} C_{zy}^{\mu\lambda},
\]

where \( B_{ji}^{\mu\lambda} = B_j^i B_i^j \) and \( C_{zy}^{\mu\lambda} = C_z^i C_y^i \).

If the transform by \( \phi^\kappa_\lambda \) of any normal vector to \( M^n \) is orthogonal to itself, the submanifold \( M^n \) is called anti-holomorphic in \( M^{2m+1} \). Since the rank of \( \phi^\kappa_\lambda \) is \( 2m \), we have \( 2m + 1 - n - 1 \leq n \), that is, \( m \leq n \).

For an anti-holomorphic submanifold \( M^n \) in \( M^{2m+1} \), we have equations of the form

\[
\begin{align*}
\phi^\kappa_\lambda B^\lambda_i &= f^i_k B^k_h - f^i_z C^z_x, \\
\phi^\kappa_z C^\lambda_y &= f^i_y B^i_z, \\
\xi^\kappa &= \xi^i_i B^i_z + \xi^z_z C^z_x.
\end{align*}
\]

Using \( \phi^\mu_\mu = -\phi^\mu_\mu \) we have, from (2.3) and (2.4),

\[
f_{ix} = f^i_x g_{xz}, \quad f_{ix} = f^i_x g_{zi} \quad \text{and} \quad f_{ij} = f^i_y g_{yj}.
\]

Applying \( \phi \) to (2.3), (2.4) and (2.5) and using (1.1) and these equations we find

\[
\begin{align*}
(i) \quad f^i_k f^j_x &= \delta^i_k - \xi^i_k \xi^j_x + f^i_k f^j_z, \\
(ii) \quad f^i_y f^j_z &= \delta^i_y - \xi^i_y \xi^j_x, \\
(iii) \quad f^i_z f^j_h &= -\xi^i_z \xi^j_h, \\
(iv) \quad f^i_k f^j_h &= \xi^i_k \xi^j_h, \\
(v) \quad f^i_z f^j_h &= -f^i_z f^j_h, \\
(vi) \quad f^i_z \xi^j_x &= 0, \\
(vii) \quad \xi^i \xi^i + \xi^x \xi^x &= 1.
\end{align*}
\]

where \( \xi^i = g_{\mu\lambda} \delta^i_\lambda \) and \( \xi^x = g_{xy} \eta^y \), (vii) be a direct consequence of \( \xi^i \xi^i = 1 \).

Differentiating (2.3), (2.4) and (2.5) covariantly along \( M^n \) and using (1.2), (2.7), equations of Gauss and those of Weingarten

\[
\begin{align*}
\nabla_j B^z_i &= h^z_{ji} C^x_x, \\
\nabla_j C^z_x &= -h^z_j B^x_i,
\end{align*}
\]

where \( \nabla_j \) denotes the operator of covariant differentiation along \( M^n \) and \( h^z_{ji} \) and \( h^i_j = h^z_{ji} g_{xz} \) are the second fundamental tensors of \( M^n \) with respect to normals.
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\[ C^\xi \text{ we find} \]

\[
\begin{align*}
(i) \quad \nabla_j f^i_f^h &= \delta^h_j \xi^i - g^h_{ji} f^h_x - h^h_{j} f^x_{h} f^i_x, \\
(ii) \quad \nabla_j \tilde{f}^x_i &= g^h_{ji} \xi^x + h^x_{j} f^i_x, \\
(iii) \quad \nabla_j f^i_y &= \delta^i_j \xi^y - h^y_{j} f^i_x, \\
(iv) \quad h^x_{j} f^i_y &= h^y_{j} f^i_x, \\
(v) \quad \nabla_j \xi^y = h^y_{j} \xi^x + f^i_y, \\
(vi) \quad \nabla_j \xi^x = - h^x_{j} \xi^y. 
\end{align*}
\]

(2.8)

I. The case in which \( \xi^\xi \) is tangent to \( M^n \).

Now suppose that \( \xi^\xi \) is tangent to \( M^n \), that is, \( \xi^\xi = 0 \). From (2.7), (i) and (ii) we find

\[-f^i_j f^j_i = 2(n+1-n).\]

Thus, if \( n=m+1 \), we have \( f^i_j = 0 \), and (2.7) and (2.8) respectively reduces to

\[
\begin{align*}
(i) \quad f^x_i f^j_x &= \delta^j_i - \xi^j_i \xi^i, \\
(ii) \quad f^i_j f^j_x &= \delta^x_x, \\
(iii) \quad \xi^x_i \xi^i = 0, \\
(iv) \quad \xi^i_i \xi^i = 1 
\end{align*}
\]

(2.9)

and

\[
\begin{align*}
(i) \quad \delta^h_j \xi^i - g^h_{ji} \xi^h + h^h_{j} \xi^h - h^h_{j} f^x_{h} f^i_x &= 0, \\
(ii) \quad \nabla_j f^x_i &= 0, \\
(iii) \quad h^h_{j} f^i_y &= h^y_{j} f^i_x, \\
(iv) \quad \nabla_j \xi^y &= 0, \\
v) \quad \xi^x_i \xi^i &= 0. 
\end{align*}
\]

(2.10)

Equation (2.10), (i) shows that an anti-holomorphic submanifold tangent to \( \xi^\xi \) cannot be totally umbilical or totally contact umbilical. Because if \( h^h_{ji} \) is of the from \( (\alpha g^h_{ji} + \beta \xi^j_i \xi^i) h^x \), then from (2.10), (i) we have

\[(n-1) \xi^i = (n-1) \alpha h^x \xi^x + \beta h^x f^x_i, \]

and consequently, transvecting with \( \xi^i \) and using (2.9), (iii) gives \( (n-1) \xi^i \xi^i = 0 \), which is a contradiction for \( n>1 \).

From (2.10), (ii) and the Ricci identity we find

\[ K^h_{kji} f^x_h = K^x_{kji} f^x_h, \]

(2.11)

where \( K^h_{kji} \) is the curvature tensor of \( M^n \) and \( K^x_{kji} \) that of the normal bundle of \( M^n \).

Taking account of (2.9), (i), (ii) and (2.11) yields

\[ K^h_{kji} = K^x_{kji} f^x_h, K^x_{kji} = K^h_{kji} f^x_h. \]

(2.12)
with the help of $K_{kji} h^i = 0$. Equations (2.12) shows that $K_{kji} h = 0$ and $K_{kji} z = 0$ are equivalent to each other.

II. The case in which $\xi^x$ is normal to $M^n$.

From (2.7), (i), (ii) and using (2.7), (vii) we have

$$2\xi^i \xi^j + f_{ij} f^{ij} = -2(m-n).$$

Suppose $n=m$ we have $f_{ij} = 0$ and $\xi^i = 0$, that is, $\xi^x$ is normal to $M^n$. Then (2.7) and (2.8) respectively become

$$2\xi^i \xi^j = -2(m-n).$$

and

$$2\xi^i \xi^j = -2(m-n).$$

Suppose that $M^n$ is totally umbilical and put $h_{ji} = g_{ji} h^i$. Then from (2.14), (i) we have

$$g_{ji} h^i = \delta_j^i h \xi,$$

from which $f^h_x = 0$, for $n>1$. From (2.14), (iv) we have

$$h^i \xi = h \xi,$$

from which transvecting with $h \xi$ and using $f_{ij} \xi = 0$ and consequently $h \xi \xi = 0$, that is, $h \xi = 0$. Thus $M^n$ must be totally geodesic.

From (2.14), (ii) and (vi), we find

$$\nabla_j \nabla_i \xi^x = -g_{ji} \xi^x,$$

from which, using the Ricci identity,

$$K_{kji} \xi^i = 0.$$

On the other hand, from (2.14), (ii) and (vi), we have

$$-K_{kji} h^i + K_{kji} h^i = -f^i_k g_{ji} + k^i_{ji} \xi^x q_{hi}.$$
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\begin{equation}
K_{kji} = K_{kxy} f_i^y f_j^x + \phi_k^g g_{ji} - \phi_j^g g_{ki} \tag{2.15}
\end{equation}

and, using taking account of \(K_{kxy} \xi^x = 0\) and (2.13), (ii),

\begin{equation}
K_{kxy} = K_{kji} f_i^y f_j^x + f_{xy} f_j^x - f_{xy} f_k^x \tag{2.16}
\end{equation}

Equation (2.15) and (2.16) mean that \(M^u\) is of constant curvature 1 if and only if the connection induced in the normal bundle is of zero curvature.

3. Proofs of the main theorems

We first of all remember that the equations of Gauss, Codazzi and Ricci are respectively

\begin{equation}
\begin{aligned}
K_{kji} &= K_{\mu\lambda} B^{\mu\lambda}_{kji} + h_{hxs} h_{hi} = h_{iha} h_{ki} \\
0 &= K_{\mu\lambda} B^{\mu\lambda}_{kji} C_{y} - (\nabla_k h_{jiy} - \nabla_j h_{kxy}), \\
K_{kxy} &= K_{\mu\lambda} B^{\mu\lambda}_{kji} C_{y} - (h_{t}^i y h_{ji} - h_{y}^i y h_{ktx})
\end{aligned}
\end{equation}

where \(K_{\mu\lambda}\), \(K_{kji}\) and \(K_{kxy}\) are covariant components of the curvature tensors of \(M^{2m+1}\), \(M^u\) and the normal bundle respectively.

\(B^{\mu\lambda}_{kji} = B^\mu_k B^\nu_j B^\lambda_i\) and \(B^{\mu\lambda}_{kxy} = B^\nu_k B^\mu_j B^\lambda_i\).

We assume that the C-Bochner curvature tensor of \(M^{2m+1}\) vanishes identically. Then from (1.6), we have

\begin{equation}
K_{\mu\lambda} + (g_{\mu\lambda} - \delta_{\mu\lambda} \delta_{xy}) L_{\mu\lambda} - (g_{\mu\lambda} - \delta_{\mu\lambda} \delta_{xy}) L_{\mu\lambda}
\end{equation}

\begin{equation}
+ L_{\mu\lambda} (g_{\mu\lambda} - \delta_{\mu\lambda} \delta_{xy}) L_{\mu\lambda}
\end{equation}

\begin{equation}
+ \phi_{\mu\lambda} M_{\mu\lambda} - \phi_{\mu\lambda} M_{\mu\lambda} + M_{\mu\lambda} \phi_{\mu\lambda} - M_{\mu\lambda} \phi_{\mu\lambda} - 2(\phi_{\mu\lambda} M_{\mu\lambda} + M_{\mu\lambda} \phi_{\mu\lambda})
\end{equation}

\begin{equation}
+ (\phi_{\mu\lambda} \phi_{\mu\lambda} - \phi_{\mu\lambda} \phi_{\mu\lambda} - 2 \phi_{\mu\lambda} \phi_{\mu\lambda}) = 0,
\end{equation}

from which, using \(g_{\mu\lambda} B^{\mu\lambda}_{ji} = g_{ji}, \phi_{\mu\lambda} B^{\mu\lambda}_{ji} = f_{ji}, \phi_{\mu\lambda} B^{\mu\lambda}_{ji} C_{xy} = -f_{xy}, \phi_{\mu\lambda} C_{xy} = 0, \delta_{\mu} B^{\mu}_{h} = \delta_h, \delta_{\mu} C_{xy} = \delta_x, \delta_{\mu} C_{xy} = \delta_y\), we find

\begin{equation}
(K_{\mu\lambda} B^{\mu\lambda}_{kji} + (g_{\mu\lambda} - \delta_{\mu\lambda} \delta_{xy}) L_{ji} - (g_{\mu\lambda} - \delta_{\mu\lambda} \delta_{xy}) L_{hi} + L_{hh} (g_{ji} - \delta_{ji} \delta_{xy}))
\end{equation}

\begin{equation}
-L_{jh} (g_{hi} - \delta_{hi} \delta_{xy}) + f_{jh} M_{ji} - f_{jh} M_{hi} + M_{hh} f_{ji} - M_{hh} f_{ki}
\end{equation}

\begin{equation}
-2(f_{kj} M_{ih} + M_{kj} f_{ih}) + (f_{hh} f_{ji} - f_{jh} f_{ki} - 2 f_{kj} f_{ih}) = 0,
\end{equation}

\begin{equation}
(K_{\mu\lambda} B^{\mu\lambda}_{kxy} C_{y} - \delta_{xy} - \delta_{xy} L_{xy} + \delta_{xy} L_{xy} + \delta_{xy} L_{xy} + \delta_{xy} L_{xy} - f_{xy} M_{xy})
\end{equation}

\begin{equation}
+ f_{jy} M_{hx} - M_{hx} f_{jy} + M_{hx} f_{ky} - 2 f_{kj} f_{xy} + (f_{hx} f_{xy} - f_{jh} f_{xy}) = 0.
\end{equation}
where $L_{ji}=L_{\mu\lambda}B_{ji}^{\mu\lambda}$, $L_{ky}=L_{\mu\lambda}B_{k}^{\mu\lambda}C_{y}^{\lambda}$, $M_{ji}=M_{\mu\lambda}B_{ji}^{\mu\lambda}=-L_{jy}f_{i}^{y}+L_{jx}f_{i}^{x}$, $M_{ky}=M_{\mu\lambda}B_{k}^{\mu\lambda}C_{y}^{\lambda}=-L_{ki}f_{y}^{i}$ and $M_{yx}=M_{\mu\lambda}C_{yx}^{\mu\lambda}=-L_{yi}L_{x}^{i}$.

Thus equations (3.3) can respectively be written as

$$
\begin{align*}
K_{kji} + (g_{kh} - \xi_{k}^{h})L_{ji} - (g_{jh} - \xi_{j}^{h})L_{ki} + L_{kh} (g_{ji} - \xi_{j}^{i}) &= 0, \\
-2(f_{kj}M_{ih} + M_{kj}f_{ih}) + (f_{kh}f_{ji} - f_{j}f_{ki} - 2f_{k}f_{hi}) &= 0,
\end{align*}
$$

(3.4)

$$
\begin{align*}
K_{kjyx} - (\xi_{j}^{y}M_{jy} - \xi_{y}^{j}L_{jy} - \xi_{j}^{y}L_{jy})^{k} = - (\xi_{j}^{y}L_{jy} - \xi_{y}^{j}L_{jy})^{k} - f_{kj}M_{jy} + f_{jy}M_{kj} - M_{kj}M_{jy} + (f_{kj}f_{jy} - f_{jy}f_{kj}) + (h_{k}^{i}h_{j}^{y} - h_{j}^{i}h_{k}^{y})
\end{align*}
$$

We now consider the case in which the vector field $\xi^{k}$ is tangent to $M^{h}$.

We now assume that the second fundamental tensors are commutative. Then from (2.12) and (3.5), we have

$$
\begin{align*}
K_{kji} + (g_{kh} - \xi_{k}^{h})M_{jy} + (g_{jh} - \xi_{j}^{h})M_{k} + (g_{jh} - \xi_{j}^{h})(g_{ki} - \xi_{k}^{i}) - (g_{kh} - \xi_{k}^{h})(g_{ji} - \xi_{j}^{i}) &= 0,
\end{align*}
$$

(3.6)

Now since the vector field $\xi^{h}$ is parallel, the Riemannian manifold $M^{h}$ is loc-
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\[ M^1 \] generated by \( \xi^h \) and \( M^{n-1} \) is totally geodesic in \( M^n \). We represent \( M^{n-1} \) in \( M^n \) by parametric equations \( y^h = y^h(x^a) \) \((a, b, c, \ldots = 1, 2, \ldots, (n-1))\) and put \( B^k_b = \frac{\partial y^k}{\partial x^b} \). Then we have \( \xi_i B^i_b = 0 \) and the curvature tensor \( K_{dca} \) of \( M^{n-1} \) is given by

\[ K_{dca} = K_{klij} B^{klij}_d \]

Thus transvecting \((3.6)\) with \( B^{klij}_d \), we obtain

\[ K_{dca} + g_{da} C_{cb} - g_{ca} C_{db} + C_{da} g_{cb} - C_{ca} g_{db} = 0, \]

where \( g_{cb} = g_{ji} B^i_c B^j_b \) is the metric tensor of \( M^{n-1} \) and

\[ C_{cb} = N_{lj} B^l_c B^j_b + \frac{1}{2} g_{cb} \]

Equation \((3.8)\) shows that the Weyl conformal curvature tensor of \( M^{n-1} \) vanishes and \( M^{n-1} \) is conformally flat if \( n-1 \geq 4 \). Thus we have completely proved Theorem 1.

II. The case in which the vector field \( \xi^k \) is normal to \( M^n \)

We now assume that \( n=m \). Then the vector field \( \xi^k \) is normal to \( M^n \) and \( f_{ji} = 0 \). Then from the first equation of \((3.4)\) we have

\[ K_{klij} + g_{kh} L_{ji} - g_{jh} L_{ki} + L_{kh} g_{ji} - L_{jh} g_{ki} - (h_{kh} h_{ji} - h_{jh} h_{ki} x^i) = 0. \]

If \( M^n \) is umbilical, that is, \( h_{ji} = g_{ji} x^i \), then we can write \((3.9)\) in the form

\[ K_{klij} + g_{kh} D_{ji} - g_{jh} D_{ki} + g_{ji} D_{kh} - g_{ki} D_{jh} = 0, \]

where \( D_{ji} = L_{ji} - \frac{1}{2} h_{ji} x^i g_{ji} \).

Equation \((3.10)\) shows that the Weyl conformal curvature tensor of \( M^n \) vanishes. Thus we have completely proved Theorem 2.

We next obtain from the second equation of \((3.4)\)

\[ K_{klij} + f_{kx} f_{ji} - f_{jx} M_{ki} + f_{ki} M_{jx} - f_{ij} M_{kx} + (f_{kx} f_{ji} - f_{jx} f_{ki}) + (h_{k}^i h_{jx}^t - h_{j}^i h_{kx}^t) = 0. \]

If the second fundamental tensors of \( M^n \) commute, then we have from \((3.11)\)

\[ K_{klij} - f_{kx} M_{ji} + f_{jx} M_{ki} - M_{ki} f_{ji} + M_{ji} f_{ki} + (f_{kx} f_{ji} - f_{jx} f_{ki}) = 0, \]
from which, by transvecting with $f^x_i f^x_h$ and using (2.13), (i)

\[(3.13)\]

$$K_{kjiw} f^y_i f^x_h - g_{kh} M_{jy} f^y_i + g_{ji} M_{ky} f^y_i - M_{ky} f^y_h g_{ji} + M_{ji} f^y_h g_{hi} = 0.$$ 

Substituting (3.13) into (2.15), we find

\[(3.14)\]

$$K_{kji} - g_{kh} M_{jy} f^y_i + g_{ji} M_{ky} f^y_i = 0,$$

which shows that the Weyl conformal curvature tensor of $M^n$ vanishes. Thus we have completely proved Theorem 3.

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REFERENCES


