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SOME NEW COMPACTNESS CHARACTERIZATIONS FROM GRAPHS

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1. Introduction and preliminaries

In this paper, if X, Y, and Z are topological spaces and $\phi: X\to Z$ and $\alpha: Y\to Z$ are functions, we denote $\{(x, y)\in X\times Y: \phi(x)=\alpha(y)\}$ by $E(\phi, \alpha, X\times Y, Z)$. The following result is well-known (e.g. see [5] for a more general result).

THEOREM 1.1 *If Z is Hausdorff, then $E(\phi, \phi, X\times X, Z)$ is a closed subset of $X\times X$ for each space X and continuous function $\phi: X\to Z$.*

The results in Theorem 1.2 motivate our main results. In a space, we let $\Sigma(x)$ represent the collection of open subsets which have x as an element and $\mathrm{cl}(K)$ represent the closure of a subset K of the space.

THEOREM 1.2. *The following statements are equivalent for a space Z:*

(a) Z is Hausdorff.

(b) $E(\phi, \alpha, X\times Y, Z)$ is closed in $X\times Y$ for any two spaces X, Y and continuous functions $\phi: X\to Z$ and $\alpha: Y\to Z$.

(c) $E(\phi, \alpha, X\times X, Z)$ is a closed subset of $X\times X$ for any space X and continuous functions $\phi, \alpha: X\to Z$.

PROOF. Proof that (a) implies (b). Suppose Z is Hausdorff, let X and Y be any spaces, let $\phi: X\to Z$ and $\alpha: Y\to Z$ be continuous and let $(p, q)\in\mathrm{cl}(E(\phi, \alpha, X\times Y, Z))$. There is as filterbase Ω on $E(\phi, \alpha, X\times Y, Z)$ with $\Omega\to(p, q)$. If π_x and π_y represent the projections of $X\times Y$ onto X and Y, respectively, then $\pi_x(\Omega)\to p$, $\pi_y(\Omega)\to q$ and for each $F\in\Omega$ we have $\phi(\pi_x(F))=\alpha(\pi_y(F))$. So $\phi(\pi_x(\Omega))\to\phi(p)$ and $\alpha(\pi_y(\Omega))\to\alpha(q)$. Since $\phi(\pi_x(\Omega))=\alpha(\pi_y(\Omega))$ and Z is Hausdorff, we have $\phi(p)=\alpha(q)$. The proof that (a) implies (b) is complete.

Proof that (b) implies (c). Obvious.

Proof that (c) implies (a). Suppose Z is not Hausdorff.

Choose x_0, $z_0\in Z$ with $x_0\neq z_0$ such that $\Omega=\{V\cap W: V\in\Sigma(x_0), W\in\Sigma(z_0)\}$ is a filterbase on Z. Let $Z(x_0, \Omega)$ be Z with $\{A\subset Z: x_0\notin A$ or $F\subset A$ for some $F\in\Omega\}$

as its topology. Let $\phi: Z(x_0, \Omega)\to Z$ be the identity function and let $\alpha: Z(x_0, \Omega)\to Z$ be defined by $\alpha(x)=x$ if $x\neq x_0$ and $\alpha(x_0)=z_0$. It is evident that both ϕ and α are continuous at any $x\neq x_0$. If $W\in\Sigma(\phi(x_0))$ ($W\in\Sigma(\alpha(x_0))$) then $W\cup\{x_0\}\in\Sigma(x_0)$ in $Z(x_0, \Omega)$ and $\phi(W\cup\{x_0\})=W$ ($\alpha(W\cup\{x_0\})\subset W$). Thus both ϕ and α are continuous at x_0. However, $(x_0, x_0)\in \mathrm{cl}(E(\phi, \alpha, Z(x_0, \Omega)\times Z(x_0, \Omega), Z))$ and $(x_0, x_0)\notin E(\phi, \alpha, Z(x_0, \Omega)\times Z(x_0, \Omega), Z)$, so Z does not satisfy condition (c). The proof of the Theorem is complete.

It is interesting to note that formulations which are strikingly similar to those of Theorem 1.2 characterize compactness. These formulations are our main results and are presented in section 2. If X and Y are sets we will let $\mathscr{C}(X, Y)$ represent the class of functions (or bijections) from X to Y. If $\phi\in\mathscr{C}(X, Y)$ we will represent the graph of ϕ(i.e. $\{(x, \phi(x)): x\in X\}$) by $G(\phi)$. If X, Y are spaces and $\phi\in\mathscr{C}(X, Y)$ we say that ϕ has a *closed graph* if $G(\phi)$ is a closed subset of $X\times Y$. If X is a space and Ω is a filterbase on X we let $\mathrm{ad}\Omega$ denote the adherence of Ω. The following theorem, asserted here without proof. motivates Definition 1.4 which extends the notion of closed graph.

THEOREM 1.3. *If X and Y are spaces, $\phi\in\mathscr{C}(X, Y)$ has a closed graph if and only if $\{\phi(x)\}$ is a closed subset of Y and $\mathrm{ad}\phi(\Omega)\subset\{\phi(x)\}$ for each $x\in X$ and filterbase Ω on $X-\{x\}$ with $\Omega\to x$.*

DEFINITION 1.4. If X and Y are spaces then $\phi\in\mathscr{C}(X, Y)$ has a *subclosed graph* if $\mathrm{ad}\phi(\Omega)\subset\{\phi(x)\}$ for each $x\in X$ and filterbase Ω on $X-\{x\}$ with $\Omega\to x$.

We let R represent the class of topological spaces or any class of spaces containing the Hausdorff completely normal spaces. If X is a set and $x_0\in X$ and Ω is a filterbase on X, then $X(x_0, \Omega)=\{A\subset X: x_0\notin A$ or $F\subset A$ for some $F\in\Omega\}$ is a topology on X and the following result is readily verified.

THEOREM 1.5. *If x_0 is an element of a set X and Ω is a filterbase on X with empty intersection on $X-\{x_0\}$, then $X(x_0, \Omega)$ is in class R.*

2. Main results

Theorems 2.1 and 2.2 are our main results.

THEOREM 2.1. *The following statements are equivalent for a space Z:*
(a) Z is compact.

(*b*) *$E(\phi, \alpha, X\times Y, Z)$ is a closed subset of $X\times Y$ for any two spaces X, Y in R and $\phi\in\mathscr{C}(X, Z)$, $\alpha\in\mathscr{C}(Y, Z)$ with subclosed graphs.*

(*c*) *$E(\phi, \alpha, X\times X, Z)$ is a closed subset of $X\times X$ for any space X in R and ϕ, $\alpha\in\mathscr{C}(X, Z)$ with subclosed graphs.*

PROOF. Let Z be compact, X, Y be any sapces, $\phi\in\mathscr{C}(X, Z)$ $\alpha\in\mathscr{C}(Y, Z)$ with subclosed graphs, and let (p, q) be a limit point of $E(\phi, \alpha, X\times Y, Z)$. There is a filterbase Ω on $E(\phi, \alpha, X\times Y, Z)-\{(p, q)\}$ with $\Omega\rightarrow(p, q)$. If $\pi_x(F)=\{p\}(\pi_y(F)=\{q\})$ for some $F\in\Omega$, then we may assume, without loss, that $\pi_y(\Omega)$ $(\pi_x(\Omega))$ is a filterbase on $Y-\{q\}$ $(X-\{p\})$. Since $\pi_y(\Omega)\rightarrow q$ $(\pi_x(\Omega)\rightarrow p)$ we have $\mathrm{ad}\alpha(\pi_y(\Omega))\subset\{\alpha(q)\}(\mathrm{ad}\phi(\pi_x(\Omega))\subset\{\phi(p)\})$. We see readily that $\{\phi(p)\}\subset \mathrm{ad}\,\phi(\pi_x(\Omega))=\mathrm{ad}\alpha(\pi_y(\Omega))\subset\{\alpha(q)\}(\{\alpha(q)\}\subset\mathrm{ad}\alpha(\pi_y(\Omega))=\mathrm{ad}\phi(\pi_x(\Omega))\subset\{\phi(p)\})$. So $\phi(p)=\alpha(q)$ and $(p, q)\in E(\phi, \alpha, X\times Y, Z)$. Otherwise, without loss, we may assume that $\pi_x(\Omega)$ and $\pi_y(\Omega)$ are filterbases on $X-\{p\}$ and $Y-\{q\}$, respectively; since $\pi_x(\Omega)\rightarrow p$ and $\pi_y(\Omega)\rightarrow q$ and Z is compact we have $\phi\neq\mathrm{ad}\alpha(\pi_y(\Omega))\subset\{\alpha(q)\}$. Since $\mathrm{ad}\alpha(\pi_y(\Omega))=\mathrm{ad}\phi(\pi_x(\Omega))\subset\{\phi(p)\}$ we get $\phi(p)=\alpha(q)$ and $(p, q)\in E(\phi, \alpha, X\times Y, Z)$. The proof that (*a*) implies (*b*) is complete.

Now it is clear that if (*b*) is satisfied when $\mathscr{C}(X, Z)$ and $\mathscr{C}(Y, Z)$ are restricted to the classes of bijections, and for all X, Y in R then (*c*) is also obviously satisfied under these conditions. To complete the proof of the theorem we will show that if (*c*) is satisfied under these conditions then Z is compact. Suppose Z is not compact and let Ω be a filterbase on Z with $\mathrm{ad}\Omega=\phi$. Choose x_0, $z_0\in Z$ with $x_0\neq z_0$ and assume, without loss, that Ω is a filterbase on $Z-\{x_0, z_0\}$. Let $\phi: Z(x_0, \Omega)\rightarrow Z$ be the identity function and define $\alpha: Z(x_0, \Omega)\rightarrow Z$ by $\alpha(x_0)=z_0$, $\alpha(z_0)=x_0$ and $\alpha(x)=x$ otherwise. ϕ and α are clearly bijections; we show that $G(\phi)$ and $G(\alpha)$ are subclosed and that $(x_0, x_0)\in\mathrm{cl}(E(\phi, \alpha, Z(x_0, \Omega)\times Z(x_0, \Omega), Z)$. It is clear that $(x_0, x_0)\notin E(\phi, \alpha, Z(x_0, \Omega)\times Z(x_0, \Omega), Z)$. This will complete the proof. $G(\phi)$ $(G(\alpha))$ is subclosed. Let $x\in X$ and let Ω^* be a filter base on $Z-\{x\}$ with $\Omega^*\rightarrow x$ in $Z(x_0, \Omega)$. Then $x=x_0$ and Ω^* is stronger than Ω so $\mathrm{ad}\phi(\Omega^*)=\mathrm{ad}\Omega^*\subset\mathrm{ad}\Omega=\phi$ $(\mathrm{ad}\alpha(\Omega^*)=\mathrm{ad}\Omega^*\subset\mathrm{ad}\Omega=\phi)$. This completes the demonstration that $G(\phi)$ $(G(\alpha))$ is subclosed.

Finally, if F, $F^*\in\Omega$, let $p\in F\cap F^*$. Then $(p, p)\in(F\cup\{x_0\})\times(F^*\cup\{x_0\})$ and $(p, p)\in E(\phi, \alpha, Z(x_0, \Omega)\times Z(x_0, \Omega), Z)$; thus $(x_0, x_0)\in\mathrm{cl}(E(\phi, \alpha, Z(x_0, \Omega)\times Z(x_0, \Omega), Z))$.

The proof of the theorem is complete.

THEOREM 2.2. *A space Z is compact if and only if for each space X, $E(\phi, \phi, X\times X, Z)$ is closed in $X\times X$ for each $\phi\in\mathscr{O}(X, Z)$ with a subclosed graph.*

PROOF. *Necessity.* This follows immediately from equiealence (*c*) of Theorem 2.1.

Sufficiency. Suppose Ω is a filterbase on the space Z with $\mathrm{ad}\Omega=\phi$. We may choose x_0, $z_0\in Z$ with $x_0\neq z_0$ and assume, without loss, that Ω is a filterbase on $Z-\{x_0, z_0\}$. Let X be Z with $\{A\subset X: A\cap\{x_0, z_0\}=\phi$ or $F\subset A$ for some $F\in\Omega\}$ as the topology. Let $\phi: X\rightarrow Y$ be defined by $\phi(x)=x$ if $x\notin\{x_0, z_0\}$, $\phi(x_0)=z_0$, and $\phi(z_0)=x_0$. Now, let $x\in X$ and let Ω^* be a filterbase on $X-\{x\}$ with $\Omega^*\rightarrow x$ in X. Then $x\in\{x_0, z_0\}$ and we may assume, without loss, that Ω^* is stronger than Ω. So $\mathrm{ad}\phi(\Omega^*)\subset\mathrm{ad}\Omega=\phi$, and $G(\phi)$ is subclosed. Now $\phi(x_0)\neq\phi(z_0)$ so $(x_0, z_0)\notin E(\phi, \phi, X\times X, Z)$. However, $(x_0, z_0)\in\mathrm{cl}(E(\phi, \phi, X\times X, Z))$

The proof of the the theorem is complete.

We close by observing that other characterizations of T_1 compact spaces from graphs may be found in [1], [2], [3], [4], and [6].

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