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ON EXTENDED TOPOLOGIES

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1. Introduction

The concept of a simple extension of a topology was first introduced by Norman Levine [2]. We may state his definition of a simple extension of a topology as follows: Let $(X, \mathscr{T})$ be a topological space where $\mathscr{T}$ is some topology on X, and let A be a subset of X such that $A \notin \mathscr{T}$. Then the topology $\mathscr{T}(A)=\{U\cup(U'\cap A)|U,\ U'\in\mathscr{T}\}$ is called *a simple extension* of $\mathscr{T}$. It is the purpose of this paper to introduce a concept of extending $\mathscr{T}$ by an arbitrary number of subsets of X, to give some theorems concerned with extended topologies, and to consider the idea of extending one topology by the members of an open basis of another topology.

2. Arbitrary extensions of $\mathscr{T}$

Let $(X, \mathscr{T})$ be a topological space and let A and B be subsets of X which are respectively not $\mathscr{T}$-open and not $\mathscr{T}(A)$-open. We will let $\mathscr{T}(A)(B)$ denote a simple extension of a simple extension topology; i.e., the simple extension topology $\mathscr{T}(A)$ is extended by B. Let $\mathscr{T}(A,B)$ denote the extension of $\mathscr{T}$ by A and B simultaneously; i.e., $\mathscr{T}(A,B)=\{N\cup(O\cap A)\cup(O'\cap B)\cup(G\cap A\cap B)|\ N,O,O',G\in\mathscr{T}\}$. Clearly $\mathscr{T}(A)(B)=\mathscr{T}(A,B)$ because $\mathscr{T}(A)(B)=\{[O\cup(A\cap O')]\cup(B\cap[M\cup(A\cap M')])|O,\ O',\ M,\ M'\in\mathscr{T}\}=\{O\cup(A\cap O')\cup(B\cap M)\cup(B\cap A\cap M')|O,\ O',\ M,\ M'\in\mathscr{T}\}$.

Let $G=\{A_\alpha|A_\alpha\subset X,\ A_\alpha\notin\mathscr{T},\ \alpha\in\Lambda\}$ and $G'=\{C_\omega=\cap\{A_\alpha|\alpha\in\Lambda'\}\,|\,\Lambda'$ a finite subcollection of Λ; $\cap\{A_\alpha|\alpha\in\Lambda'\}$ not expressible in the form $\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda\}\cup U$; $U_\alpha,\ U\in\mathscr{T}$; $\omega\in W\}$.

Then the extension of $\mathscr{T}$ by all the members of G (i.e., the extension of $\mathscr{T}$ by every member of G simultaneously) is denoted $\mathscr{T}[A_\alpha]$ and is defined to be the collection

$$\{\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda\}\cup[\cup\{U_\omega\cap C_\omega|\omega\in W\}]\cup U|U,U_\alpha,U_\omega\in\mathscr{T}\ ;\ \alpha\in\Lambda\ ;\ \omega\in W\}.$$

Set $G\cup G'=\{B_\beta|B_\beta\in G$ or $B_\beta\in G',\ \beta\in(\Lambda\cup W)\}$. Then $\mathscr{T}[A_\alpha]=\mathscr{T}[B_\beta]$ since $\mathscr{T}[A_\alpha]=\{\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda\}\cup[\cup\{U_\omega\cap C_\omega|\omega\in W\}]\cup U|\ U_\alpha,\ U,\ U_\omega\in\mathscr{T}\ ;\alpha\in\Lambda\ ;$

$\omega\in W\}$ and every $\mathscr{T}[B_\beta]$-open set may be expressed in the form $\cup\{V_\beta\cap B_\beta|\beta\in(\Lambda\cup W)\}\cup V$ where V_β, $V\in\mathscr{T}$, $\beta\in(\Lambda\cup W)$. Therefore in the future (unless otherwise stated) we shall assume that an arbitrary extension $\mathscr{T}[D_\delta]$ ($\mathscr{D}=\{D_\delta|D_\delta\subset X,\ D_\delta\notin\mathscr{T},\ \delta\in\Delta\}$) of $\mathscr{T}$ is one in which every $\mathscr{T}[D_\delta]$-open set is expressibe in the form $\cup\{U_\delta\cap D_\delta|\delta\in\Delta\}\cup U$ where U_δ, $U\in\mathscr{T}$, $\delta\in\Delta$.

THEOREM 1. *Let $G=\{A_\alpha|A_\alpha\subset X,\ A_\alpha\notin\mathscr{T},\ \alpha\in\Lambda\}$. Then $\mathscr{T}[A_\alpha]$ is equivalent to a well-ordered succession of simple extensions.*

PROOF. Our proof will be by transfinite induction. By Zermelo's Theorem we can well-order the collection G. Let $P_{\alpha'}$ denote the proposition that $\mathscr{T}[A_\alpha]_{\alpha\leq\alpha'}$ is a succession of simple extensions where $\alpha\leq\alpha'$ for each A_α in the extension. P_1 is trivially true since $\mathscr{T}[A_1]=\mathscr{T}(A_1)$ is a simple extension. From above P_2 is true since $\mathscr{T}[A_\alpha]_{\alpha\leq 2}=\mathscr{T}(A_1,\ A_2)=\mathscr{T}(A_1)(A_2)$. Assume that P_α is true for every α such that $\alpha<\alpha'$; i.e., assume $\mathscr{T}[A_\alpha]_{\alpha<\alpha'}$ is a succession of simple extensions. Let $K=\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda,\ \alpha\leq\alpha'\}\cup U$ be arbitrary in $\mathscr{T}[A_\alpha]_{\alpha\leq\alpha'}$ where U, $U_\alpha\in\mathscr{T}$, $\alpha\in\Lambda$. We can write $K=\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda,\ \alpha<\alpha'\}\cup(U_{\alpha'}\cap A_{\alpha'})\cup U$. Set $V=\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda,\ \alpha<\alpha'\}\cup U$. Now $V\in\mathscr{T}[A_\alpha]_{\alpha<\alpha'}$ and $K=V\cup(U_{\alpha'}\cap A_{\alpha'})$ belongs to $\mathscr{T}[A_\alpha]_{\alpha<\alpha'}(A_{\alpha'})$ since $V, U_{\alpha'}\in\mathscr{T}[A_\alpha]_{\alpha<\alpha'}$. Therefore $\mathscr{T}[A_\alpha]_{\alpha\leq\alpha'}\subset\mathscr{T}[A_\alpha]_{\alpha<\alpha'}(A_{\alpha'})$. Let $M=W\cup(W'\cap A_{\alpha'})$ be arbitrary in $\mathscr{T}[A_\alpha]_{\alpha<\alpha'}(A_{\alpha'})$ where W, $W'\in\mathscr{T}[A_\alpha]_{\alpha<\alpha'}$. We can write $W=\cup\{W_\alpha\cap A_\alpha|\alpha\in\Lambda,\ \alpha<\alpha'\}\cup O$ and $W'=\cup\{W_\alpha'\cap A_\alpha|\alpha\in\Lambda,\alpha<\alpha'\}\cup O'$ where W_α, O, W_α', $O'\in\mathscr{T}$, $\alpha\in\Lambda$. Hence $M=W\cup[\cup\{W_\alpha'\cap A_\alpha|\alpha\in\Lambda,\ \alpha<\alpha'\}\cap A_{\alpha'}]\cup(O'\cap A_{\alpha'})=W\cup[\cup\{W_\alpha'\cap A_\alpha\cap A_{\alpha'}|\alpha\in\Lambda,\ \alpha<\alpha'\}]\cup(O'\cap A_{\alpha'})$. It is obvious that W, $\cup\{W_\alpha'\cap A_\alpha\cap A_{\alpha'}|\alpha\in\Lambda,\ \alpha<\alpha'\}$, and $(O'\cap A_{\alpha'})$ each belong to $\mathscr{T}[A_\alpha]_{\alpha\leq\alpha'}$ so that $M\in\mathscr{T}[A_\alpha]_{\alpha\leq\alpha'}$. Hence $\mathscr{T}[A_\alpha]_{\alpha\leq\alpha'}=\mathscr{T}[A_\alpha]_{\alpha<\alpha'}(A_{\alpha'})$ and since $\mathscr{T}[A_\alpha]_{\alpha<\alpha'}$ is a succession of simple extensions, $\mathscr{T}[A_\alpha]_{\alpha<\alpha'}(A_{\alpha'})$ is also a succession of simple extensions. Therefore $P_{\alpha'}$ is true and this completes the proof.

3. Some properties of extended topologies.

THEOREM 2. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies for a set S. Then $\mathscr{T}_1\subset\mathscr{T}_2$ iff $\mathscr{T}_2$ is an extension of $\mathscr{T}_1$, and in this case $\mathscr{T}_2=\mathscr{T}_1[O_\alpha]$ where $G=\{O_\alpha|O_\alpha\notin\mathscr{T}_1,\ O_\alpha$ is a member of an open basis for $\mathscr{T}_2,\ \alpha\in\Lambda\}$.*

PROOF. Let $\mathscr{T}_2$ be an extension of $\mathscr{T}_1$. Then by the definition of an extension topology we have $\mathscr{T}_1 \subset \mathscr{T}_2$.

Conversely let $\mathscr{T}_1 \subset \mathscr{T}_2$. Suppose $\mathscr{U}_2$ is an open basis for $\mathscr{T}_2$. Let $\mathfrak{O} = \mathscr{U}_2 - \mathscr{T}_1 = \{O_\alpha \mid \alpha \in \Lambda\}$. Consider $\mathscr{T}_1[O_\alpha] = \{\bigcup\{U_\alpha' \cap O_\alpha \mid \alpha \in \Lambda\} \cup U' \mid U_\alpha',\ U' \in \mathscr{T}_1,\ \alpha \in \Lambda\}$. Let $A = \bigcup\{U_\alpha' \cap O_\alpha \mid \alpha \in \Lambda\} \cup U'$ be arbitrary in $\mathscr{T}_1[O_\alpha]$ where $U_\alpha',\ U' \in \mathscr{T}_1$, $\alpha \in \Lambda$. Since $\mathscr{T}_1 \subset \mathscr{T}_2$ we have $U_\alpha',\ U' \in \mathscr{T}_2$, $\alpha \in \Lambda$. Hence $(U_\alpha' \cap O_\alpha) \in \mathscr{T}_2$, $\alpha \in \Lambda$, so that $A \in \mathscr{T}_2$. Therefore $\mathscr{T}_1[O_\alpha] \subset \mathscr{T}_2$. Now let B be arbitrary in $\mathscr{T}_2$ Then $B = \bigcup\{U_\beta \mid U_\beta \in \mathscr{U}_2,\ \beta \in \Gamma\} = \bigcup\{U_\beta \mid U_\beta \in \mathfrak{O},\ \beta \in \Gamma\} \cup [\bigcup\{U_\beta \mid U_\beta \in \mathscr{T}_1,\ \beta \in \Gamma\}]$ which is of the form of an element of $\mathscr{T}_1[O_\alpha]$ since $U_\beta = U_\beta \cap S$ for $U_\beta \in \mathfrak{O}$, $\beta \in \Gamma$ and since $\bigcup\{U_\beta \mid U_\beta \in \mathscr{T}_1,\ \beta \in \Gamma\} = U' \in \mathscr{T}_1$. Thus $\mathscr{T}_2 \subset \mathscr{T}_1[O_\alpha]$ and therefore $\mathscr{T}_1[O_\alpha] = \mathscr{T}_2$.

THEOREM 3. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies for a set X, $\mathscr{F}$ an arbitrary filter on X, and A_1 and A_2 the sets of convergence points of $\mathscr{F}$ with respect to $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. If $G_2 = \{T_\alpha \mid T_\alpha \notin \mathscr{T}_1, T_\alpha$ is a member of an open basis for $\mathscr{T}_2,\ \alpha \in \Lambda\}$, then the set $C = A_1 \cap A_2$ is the set of convergence points of $\mathscr{F}$ with respect to $\mathscr{T}_3$ iff $\mathscr{T}_3 = \mathscr{T}_1[T_\alpha]$.*

PROOF. Let $\mathscr{U}_2$ be an open basis for $\mathscr{T}_2$ and set $G_2 = \{T_\alpha \mid T_\alpha \notin \mathscr{T}_1,\ T_\alpha \in \mathscr{U}_2,\ \alpha \in \Lambda\}$. Let $\mathscr{T}_3 = \mathscr{T}_1[T_\alpha]$ and let $\mathscr{F}$ be an arbitrary filter on X. Suppose p is an arbitrary point in $C = A \cap B$ where A is the set of convergence points of $\mathscr{F}$ with respect to $\mathscr{T}_1$, B is the set of convergence points of $\mathscr{F}$ with respect to $\mathscr{T}_2$. (If $C = \phi$, then $\mathscr{F}$ will converge to no point of X with respect to $\mathscr{T}_3$ since we show below that $\mathscr{F}$ converges to no point in $X - (A \cap B)$ with respect to $\mathscr{T}_3$.) Then $\mathscr{F} \to p$ with respect to $\mathscr{T}_1$, and $\mathscr{F} \to p$ with respect to $\mathscr{T}_2$. Let N_p^3 be an arbitrary $\mathscr{T}_3$-open neighborhood of p, say $N_p^3 = \bigcup\{U_\alpha' \cap T_\alpha \mid \alpha \in \Lambda\} \cup U'$ where U', $U_\alpha' \in \mathscr{T}_1$, $\alpha \in \Lambda$. Now p belongs to at least one of the members of the union which forms N_p^3. If $p \in U' \in \mathscr{T}_1$, then $U' \in \mathscr{F}$ since $\mathscr{F} \to p$ with respect to $\mathscr{T}_1$ and since U' is a $\mathscr{T}_1$-neighborhood of p. Consequently $N_p^3 \in \mathscr{F}$ since $U' \subset N_p^3$. On the other hand, if $p \in U_\alpha' \cap T_\alpha$ for some $\alpha \in \Lambda$, then $p \in U_\alpha'$ and $p \in T_\alpha$. Thus $U_\alpha' \in \mathscr{F}$ since $\mathscr{F} \to p$ with respect to $\mathscr{T}_1$, and $T_\alpha \in \mathscr{F}$ since $\mathscr{F} \to p$ with respect to $\mathscr{T}_2$. Therefore $U_\alpha' \cap T_\alpha \in \mathscr{F}$, so that $N_p^3 \in \mathscr{F}$ since $U_\alpha' \cap T_\alpha \subset N_p^3$. Since N_p^3 is an arbitrary $\mathscr{T}_3$-open neighborhood of p we see that every $\mathscr{T}_3$-neighborhood of p belongs to $\mathscr{F}$; i.e., $\mathscr{F} \to p$ with respect to $\mathscr{T}_3$. Hence, $\mathscr{F}$ converges to every point of C with respect to $\mathscr{T}_3$. We shall now show that $\mathscr{F}$ converges to no other points with respect to $\mathscr{T}_3$. Let q be an arbitrary point in $X - (A \cap B)$. Then either $\mathscr{F} \not\to q$ with respect to $\mathscr{T}_1$ or $\mathscr{F}$

$\not\to q$ with respect to $\mathscr{T}_1$. Suppose $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_1$. Then there exists a $\mathscr{T}_1$-neighborhood N_q' of q such that $N_q'\notin\mathscr{F}$. Since $N_q'\in\mathscr{T}_3$ also, N_q' is a $\mathscr{T}_3$-neighborhood of q. Thus $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_3$ if $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_1$. Now suppose $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_2$. Then there exists a $\mathscr{T}_2$-neighborhood N_q^2 of q such that $N_q^2\notin\mathscr{T}$. There exists a $\mathscr{T}_2$-open set T such that $q\in T\subset N_q^2$. If $T\in\mathscr{T}_1$ also, then $T\in\mathscr{T}_3$, and clearly $T\notin\mathscr{F}$ which implies $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_3$. If $T\notin\mathscr{T}_1$, then $T=\phi\cup(X\cap T)\in\mathscr{T}_3$, and since $T\notin\mathscr{F}$ we see that $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_3$. Therefore in either case $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_3$ if $\mathscr{F}\not\to q$ with respect to $\mathscr{T}_2$. Hence, $\mathscr{F}$ converges to no point of $X-C$.

Conversely, suppose for every filter $\mathscr{F}$ on X that $C=A\cap B$ is the set of convergence points of $\mathscr{F}$ with respect to $\mathscr{T}_3$ where A is the set of convergence points of $\mathscr{F}$ with respect to $\mathscr{T}_2$. We wish to show that $\mathscr{T}_3=\mathscr{T}_1[T_\alpha]$ where the T_α belong to G_2. Let W be arbitrary in $\mathscr{T}_3$. Let us construct, for every $p\in W$, a filter $\mathscr{F}_p=\{F\mid p\in U_p\cap T_p\subset F\subset X$ for some $\mathscr{T}_1$-open set U_p of p and for some $\mathscr{T}_2$-open set T_p of $p\}$. For each $p\in W$, let A_p be the set of convergence points of $\mathscr{F}_p$ with respect to $\mathscr{T}_1$, and let B_p be the set of convergence points of $\mathscr{F}_p$ with respect to $\mathscr{T}_2$. Then, by hypothesis, $C_p=A_p\cap B_p$ is the set of convergence points of $\mathscr{F}_p$ with respect to $\mathscr{T}_3$. Then $p\in C_p=A_p\cap B_p$ and hence, $W\in\mathscr{F}_p$ since W is a $\mathscr{T}_3$-neighborhood of p. We now prove a lemma.

LEMMA 3.1. *Let $\mathscr{F}_p$ be constructed as above, for every $p\in W$. Then $\cup\{U_p\cap T_p\mid p\in W\}\cup U=W$ for some sets $U,U_p\in\mathscr{T}_1$, $p\in W$, $U\subset W$ and some sets $T_p\in\mathscr{T}_2$, $p\in W$ such that the U_p and T_p belong to $\mathscr{F}_p$ for each $p\in W$.*

PROOF. Let p' be arbitrary in W. Then there exist $U_{p'}$, $T_{p'}$ such that $p'\in U_{p'}\cap T_{p'}\subset\cup\{U_p\cap T_p\mid p\in W\}\cup U$ where $U,U_p\in\mathscr{T}_1$, $p\in W$, $U\subset W$ and $T_p\in\mathscr{T}_2$, $p\in W$ such that the U_p and T_p belong to $\mathscr{F}_p$ for each $p\in W$. Thus $W\subset\cup\{U_p\cap T_p\mid p\in W\}\cup U$ for appropriate U_p, T_p, where $p\in W$ and $U\subset W$. These "appropriate" U_p and T_p must be chosen such that $p\in U_p\cap T_p\subset W$ for each $p\in W$. This can be done since $W\in\mathscr{F}_p$ for each $p\in W$. Then $\cup\{U_p\cap T_p\mid p\in W\}\cup U\subset W$ and therefore $\cup\{U_p\cap T_p\mid p\in W\}\cup U=W$. It is evident that for some $\mathscr{T}_3$-open sets W it might be that $U_p=\phi$ for all but finitely many $p\in W$; i.e., we might have $W=(U_{p_1}\cap T_{p_1})\cup(U_{p_2}\cup T_{p_2})\cup\cdots\cup(U_{p_n}\cap T_{p_n})\cup U$ where $p_1,p_2,\cdots,p_n$,

$\in W$.

Thus we have $\mathscr{T}_3 \subset \mathscr{T}_1[T_\alpha]$.

Now let $N = \bigcup \{U_\alpha' \cap T_\alpha | \alpha \in \Lambda\} \cup U'$ be arbitrary in $\mathscr{T}_1[T_\alpha]$ where U', $U_\alpha' \in \mathscr{T}_1, \alpha \in \Lambda$. For each point q in N set $\mathscr{F}_q = \{F | q \in W \subset F \subset X$, W some $\mathscr{T}_3$-open set containing $q\}$. Then $\mathscr{F}_q$ is a filter on X for which $\mathscr{F}_q \to q$ with respect to $\mathscr{T}_3$. If C_q is the set of convergence points of $\mathscr{F}_q$ with respect to $\mathscr{T}_3$, then by hypothesis $q \in C_q = A_q \cap B_q$ where A_q is the set of convergence points of $\mathscr{F}_q$ with respect to $\mathscr{T}_1$, and B_q is the set of convergence points of $\mathscr{F}_q$ with respect to $\mathscr{T}_2$. Hence, every $\mathscr{T}_1$-neighborhood of q belongs to $\mathscr{F}_q$, and every $\mathscr{T}_2$-neighborhood of q belongs to $\mathscr{F}_q$. Thus, in particular, $N \in \mathscr{F}_q$ since N contains either a $\mathscr{T}_1$-open set of q or a $\mathscr{T}_2$-open set of q. We now prove another lemma.

LEMMA 3.2. *Let $\mathscr{F}_q$ be constructed as above, for each $q \in N$. Then $N = \bigcup \{W_q | q \in N\}$ for some sets $W_q \in \mathscr{T}_3, q \in N$ such that the W_q belong to $\mathscr{F}_q$ for each $q \in N$.*

PROOF. Let q' be arbitrary in N. Then there exists a $\mathscr{T}_3$-open set $W_{q'}$ which belongs to $\mathscr{F}_{q'}$ such that $q' \in W_{q'} \subset \bigcup \{W_q | q \in N\}$ for certain $\mathscr{T}_3$-open sets W_q which belong to $\mathscr{F}_q$ respectively for each q in N. Therefore $N \subset \bigcup \{W_q | q \in N\}$ for certain $\mathscr{T}_3$-open sets W_q ($W_q \in \mathscr{F}_q$ for each $q \in N$) such that $W_q \subset N$ for each $q \in N$. By the way $\mathscr{F}_q$ is constructed we know there exists some $\mathscr{T}_3$-open ses W_q containing q such that $W_q \subset N$ since $N \in \mathscr{F}_q$ for each $q \in N$. Hence, $N = \bigcup \{W_q | q \in N\}$ for appropriate sets $W_q \in \mathscr{T}_3$, $q \in N$ such that the W_q belong to $\mathscr{F}_q$ for each $q \in N$.

Thus N can be expressed as a $\mathscr{T}_3$-open set and since N is arbitrary in $\mathscr{T}_1[T_\alpha]$ we have that $\mathscr{T}_1[T_\alpha] \subset \mathscr{T}_3$. Hence, by Lemmas 3.1. and 3.2 $\mathscr{T}_3 = \mathscr{T}_1[T_\alpha]$.

Obviously, in Theorem 3 we may replace G_2 by $G_1 = \{U_\beta | U_\beta \notin \mathscr{T}_2$, U_β is a member of an open basis for $\mathscr{T}_1$, $\beta \in \Gamma\}$ and $\mathscr{T}_1[T_\alpha]$ by $\mathscr{T}_2[U_\beta]$ since $\mathscr{T}_1[T_\alpha] = \mathscr{T}_2[U_\beta]$. From Kowalsky [1, p. 56] we see that $\mathscr{T}_3 = \wedge(\mathscr{T}_1, \mathscr{T}_2)$, the inf topology of $\mathscr{T}_1$ and $\mathscr{T}_2$, in the complete lattice of all topologies for X. Therefore $\mathscr{T}_3$ is actually the weakest topology which is simultaneously stronger than $\mathscr{T}_1$ and $\mathscr{T}_2$.

THEOREM 4. *Let $(X, \mathscr{T})$ be a topological space and $G = \{A_\alpha | A_\alpha \subset X$, $A_\alpha \notin$*

$\mathscr{T}$, $\alpha\in\Lambda\}$. *Then* $\mathscr{T}[A_\alpha]=\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$ *iff* $A_\alpha\cap\mathrm{cl}[\cup\{A_\alpha|\alpha\in\Lambda\}-A_\alpha]=\phi$ *for every* $\alpha\in\Lambda$.

PROOF. Suppose that $A_\alpha\cap\mathrm{cl}[\cup\{A_\alpha|\alpha\in\Lambda\}-A_\alpha]=\phi$ for every $\alpha\in\Lambda$. Let $\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda\}\cup U$ be arbitrary in $\mathscr{T}[A_\alpha]$ where U_α, $U\in\mathscr{T}$, $\alpha\in\Lambda$. Let α° be an arbitrary fixed index from Λ. Then $A_{\alpha^\circ}\cap\mathrm{cl}[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha^\circ}]=\phi$. Hence, for every point $p\in A_{\alpha^\circ}$, there exists a $\mathscr{T}$-open set V_p of p such that $V_p\cap[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha^\circ}]=\phi$. Thus there exists the $\mathscr{T}$-open set $V_{\alpha^\circ}=\cup\{V_p|p\in A_{\alpha^\circ}\}$ such that $A_{\alpha^\circ}\subset V_{\alpha^\circ}$ and $V_{\alpha^\circ}\cap[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha^\circ}]=\phi$. Set $W_{\alpha^\circ}=U_{\alpha^\circ}\cap V_{\alpha^\circ}$. Then $W_{\alpha^\circ}\cap A_{\alpha^\circ}=U_{\alpha^\circ}\cap V_{\alpha^\circ}\cap A_{\alpha^\circ}=U_{\alpha^\circ}\cap A_{\alpha^\circ}$ since $A_{\alpha^\circ}\subset V_{\alpha^\circ}$, and $W_{\alpha^\circ}\cap[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha^\circ}]=\phi$ since $W_{\alpha^\circ}\subset V_{\alpha^\circ}$. We may do this for each $\alpha\in\Lambda$. Therefore we have $\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda\}=\cup\{W_\alpha\cap A_\alpha|\alpha\in\Lambda\}$ and for each $\alpha\in\Lambda, W_\alpha\cap[\cup\{A_{\alpha'}|\alpha'\in\Lambda\}-A_\alpha]=\phi$. Now we can write $\cup\{W_\alpha\cap A_\alpha|\alpha\in\Lambda\}\cup U=[(\cup\{W_\alpha|\alpha\in\Lambda\})\cap(\cup\{A_\alpha|\alpha\in\Lambda\})]\cup U$ because if a particular $W_{\alpha'}$ inersects $\cup\{A_\alpha|\alpha\in\Lambda\}$, then this intersection is contained in $A_{\alpha'}$. Since $\cup\{W_\alpha|\alpha\in\Lambda\}$ is $\mathscr{T}$-open, say $\cup\{W_\alpha|\alpha\in\Lambda\}=W$, we see that $\cup\{U_\alpha\cap A_\alpha|\alpha\in\Lambda\}\cup U=[W\cap(\cup\{A_\alpha|\alpha\in\Lambda\})]\cup U$ which is of the form of a member of $\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$. Hence, $\mathscr{T}[A_\alpha]\subset\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$. Now let $U\cup[U'\cap(\cup\{A_\alpha|\alpha\in\Lambda\})]$ be arbitrary in $\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$ where U, $U'\in\mathscr{T}$. Then $U\cup[U'\cap(\cup\{A_\alpha|\alpha\in\Lambda\})]=U\cup[\cup\{U'\cap A_\alpha|\alpha\in\Lambda\}]$ which is of the form of a member of $\mathscr{T}[A_\alpha]$. Hence $\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})\subset\mathscr{T}[A_\alpha]$, and consequently $\mathscr{T}[A_\alpha]=\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$.

Conversely, let $\mathscr{T}[A_\alpha]=\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$. Let α' be an arbitrary index from Λ. Assume that $A_{\alpha'}\cap\mathrm{cl}[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha'}]\neq\phi$. Then there exists a point p in $A_{\alpha'}\cap\mathrm{cl}[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha'}]$ such that for every $\mathscr{T}$-open neighborhood U_p of $p, U_p\cap A_{\alpha'}\neq\phi$ and $U_p\cap[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha'}]\neq\phi$. Now $(U_p\cap A_{\alpha'})\in\mathscr{T}[A_\alpha]$, but $(U_p\cap A_{\alpha'})\in\mathscr{T}(\cup\{A_\alpha|\alpha\in\Lambda\})$ only if there exists a $\mathscr{T}$-open set V such that $(U_p\cap A_{\alpha'})=V\cap(\cup\{A_\alpha|\alpha\in\Lambda\})$. Now since $p\in U_p\cap A_{\alpha'}=V\cap(\cup\{A_\alpha|\alpha\in\Lambda\})$, V is a $\mathscr{T}$-open neighborhood of p, and therefore $V\cap[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha'}]\neq\phi$. But $V\cap[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha'}]\subset V\cap(\cup\{A_\alpha|\alpha\in\Lambda\})=(U_p\cap A_{\alpha'})$ implies $(X-A_{\alpha'})\cap A_{\alpha'}\neq\phi$, a contradiction. Thus the assumption is false and $A_{\alpha'}\cap\mathrm{cl}[\cup\{A_\alpha|\alpha\in\Lambda\}-A_{\alpha'}]=\phi$ for every $\alpha'\in\Lambda$.

THEOREM 5. *Let* $(X, \mathscr{T})$ *be a topological space and let* A *and* B *be subsets of* X *such that* A, $B\notin\mathscr{T}$. *Then* $\mathscr{T}(A)\subset\mathscr{T}(B)$ *iff* $[(\mathrm{Bdr}_{\mathscr{T}}A)\cap A]\subset[(\mathrm{Bdr}_{\mathscr{T}}B)\cap B]$ *and there exists* $U'\in\mathscr{T}$ *such that* $U'\cap B=A\cap B$.

PROOF. Assume that there exists $U' \in \mathscr{T}$ such that $U' \cap B = A \cap B$ and that $[(\mathrm{Bdr}_{\mathscr{T}} A) \cap A] \subset [(\mathrm{Bdr}_{\mathscr{T}} B) \cap B]$. In order to show that $\mathscr{T}(A) \subset \mathscr{T}(B)$ we need only show that $A \in \mathscr{T}(B)$. We may write $A = \{A - [(\mathrm{Bdr}_{\mathscr{T}} A) \cap A]\} \cup \{(\mathrm{Bdr}_{\mathscr{T}} A) \cap A\}$. But $[(\mathrm{Bdr}_{\mathscr{T}} A) \cap A] \subset [(\mathrm{Bdr}_{\mathscr{T}} B) \cap B]$ implies that $[(\mathrm{Bdr}_{\mathscr{T}} A \cap A] \subset A \cap B = U' \cap B$. Hence $A = \{A - [(\mathrm{Bdr}_{\mathscr{T}} A) \cap A]\} \cup (U' \cap B)$, and since $\{A - [(\mathrm{Bdr}_{\mathscr{T}} A) \cap A]\}$ is $\mathscr{T}$-open and $(U' \cap B)$ is $\mathscr{T}(B)$-open we have $A \in \mathscr{T}(B)$.

Conversely, assume $\mathscr{T}(A) \subset \mathscr{T}(B)$. Then there exist $U, V' \in \mathscr{T}$ such that $A = U \cup (V' \cap B) = U \cup [(U \cap V') \cap B] = U \cup [U' \cap B]$. Thus there exists $U' \in \mathscr{T}$ such that $A \cap B = U' \cap B$. Now since $A - B = U - B$ and $U \in \mathscr{T}$ we have $[(\mathrm{Bdr}_{\mathscr{T}} A) \cap A] - B = \phi$, and since $(\mathrm{Bdr}_{\mathscr{T}} A) \cap (\mathrm{Int}_{\mathscr{T}} B) = (\mathrm{Bdr}_{\mathscr{T}} U') \cap (\mathrm{Int}_{\mathscr{T}} B)$ we have $[(\mathrm{Bdr}_{\mathscr{T}} A) \cap A] \cap (\mathrm{Int}_{\mathscr{T}} B) = \phi$. Consequently $[(\mathrm{Bdr}_{\mathscr{T}} A) \cap A] \subset [(\mathrm{Bdr}_{\mathscr{T}} B) \cap B]$.

The following corollary is now immediate.

COROLLARY 5.1. *Let $(X, \mathscr{T})$ be a topological space and $A \subset X$, $B \subset X$ such that $A, B \notin \mathscr{T}$. Then $\mathscr{T}(A) = \mathscr{T}(B)$ iff $[(\mathrm{Bdr}_{\mathscr{T}} A) \cap A] = [(\mathrm{Bdr}_{\mathscr{T}} B) \cap B]$ and there exist $U', V' \in \mathscr{T}$ such that $U' \cap B = A \cap B = V' \cap A$.*

In the next theorem we have an equivalent condition to $U' \cap B = A \cap B = V' \cap A$.

THEOREM 6. *Let $(X, \mathscr{T})$ be a topological space and let A and B be subsets of X such that $A, B \notin \mathscr{T}$. If $\mathscr{T}(A) = \mathscr{T}(B)$, then the following conditions are equivalent:*

(i) $A \cap \mathrm{cl}(B - A) = \phi = B \cap \mathrm{cl}(A - B)$.

(ii) *There exist $U', V' \in \mathscr{T}$ such that $U' \cap B = A \cap B = V' \cap A$.*

PROOF. (i)⇒(ii). Since $B \cap \mathrm{cl}(A - B) = \phi$ we know that there exists a $\mathscr{T}$-open set V such that $B \subset V$ and $V \cap (A - B) = \phi$. Therefore $V \cap A = A \cap B$. Similarly $A \cap \mathrm{cl}(B - A) = \phi$ implies that there exists a $\mathscr{T}$-open set U' such that $U' \cap B = A \cap B$.

(ii)⇒(i). Since $\mathscr{T}(A) = \mathscr{T}(B)$ we have $(\mathrm{Bdr}_{\mathscr{T}} A) \cap A = (\mathrm{Bdr}_{\mathscr{T}} B(\cap B \subset A \cap B = U' \cap B$. Therefore we write $A = [(\mathrm{Bdr}_{\mathscr{T}} A) \cap A) \cup \mathrm{Int}_{\mathscr{T}} A = (U' \cap B) \cup \mathrm{Int}_{\mathscr{T}} A$, and consequently we have $A \subset (U' \cup \mathrm{Int}_{\mathscr{T}} A)$. Set $U = U' \cup \mathrm{Int}_{\mathscr{T}} A$. Now $A \cap B = U' \cap B$ implies $B - A = B - U'$. Hence there exists the $\mathscr{T}$-open set U such that $A \subset U$ and $U \cap (B - A) = \phi$. Thus $A \cap \mathrm{cl}(B - A) = \phi$. Similarly $B \cap \mathrm{cl}(A - B) = \phi$.

Corollary 5.1 can now be restated as the following theorem:

THEOREM 7. *Let $(X, \mathscr{T})$ be a topological sapce and $A \subset X$, $B \subset X$; $A, B \notin \mathscr{T}$. Then $\mathscr{T}(A) = \mathscr{T}(B)$ iff $(\mathrm{Bdr}_{\mathscr{T}} A) \cap A = (\mathrm{Bdr}_{\mathscr{T}} B) \cap B$ and $A \cap \mathrm{cl}(B - A) =$*

$\phi=B\cap cl(A-B)$.

4. Extension of a topology by the members of an open basis for another topology

Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies for a set X. If one of these topologies is a subcollection of the other, then by Theorem 2 the stronger topology is simply an extension of the weaker topology. This extension is acquired by extending the weaker topology by the members of an open basis for the stronger topology which do not belong to the weaker topology. However, if $\mathscr{T}_1 \not\subset \mathscr{T}_2$ and $\mathscr{T}_2 \not\subset \mathscr{T}_1$ we may extend one of these topologies by the members of an open basis for the other which do not belog to the former topology and get a third topology (a topology which is stronger than both $\mathscr{T}_1$ and $\mathscr{T}_2$ and not coinciding with either). It is the purpose of this section to investigate some properties this third topology might have if $\mathscr{T}_1$ and $\mathscr{T}_2$ have certain properties.

In the remainder of this section $\mathscr{T}_1$ and $\mathscr{T}_2$ shall denote topologies for the set X such that $\mathscr{T}_1 \not\subset \mathscr{T}_2$, $\mathscr{T}_2 \not\subset \mathscr{T}_1$. We shall also set $G_2=\{B_\beta | B_\beta$ is a member of an open basis for $\mathscr{T}_2$ such that $B_\beta \notin \mathscr{T}_1$, $\beta \in \Gamma\}$ and $\mathscr{T}=\mathscr{T}_1[B_\beta]$.

THEOREM 8. *Let* $(X, \mathscr{T}_1)$ *be* T_0, T_1, *or* T_2. *Then* $(X, \mathscr{T})=(X, \mathscr{T}_1[B_\beta])$ *is respectively* T_0, T_1, *or* T_2.

PROOF. In view of Theorem 1 [2, p. 23] and Theorem 1 of this paper, the proof is complete.

COROLLARY 8.1. *Let* $(X, \mathscr{T}_1)$ *be* T_0, T_1, *or* T_2 *and let* $G_1=\{A_\alpha | A_\alpha$ *is a member of an open basis for* $\mathscr{T}_1$ *such that* $A_\alpha \notin \mathscr{T}_2$, $\alpha \in \Lambda\}$. *Then* $(X, \mathscr{T})=(X, \mathscr{T}_2[A_\alpha])$ is respectively T_0, T_1, *or* T_2.

THEOREM 9. *Let* $(X, \mathscr{T}_1)$ *and* $(X, \mathscr{T}_2)$ *be regular spaces. Then* $(X, \mathscr{T})$ *is regular.*

PROOF. For a subset C of X we let $cl^1(C)$, $cl^2(C)$, $cl^*(C)$ denote respectively the $\mathscr{T}_1$-closure, the $\mathscr{T}_2$-closure, and the $\mathscr{T}$-closure of C. Let p be an arbitrary point in X and let $\cup\{U_\beta \cap B_\beta | \beta \in \Gamma\} \cup U$ (where U_β, $U \in \mathscr{T}_1$, $\beta \in \Gamma$) be an arbitrary $\mathscr{T}$-open set such that $p \in \cup\{U_\beta \cap B_\beta | \beta \in \Gamma\} \cup U$. If $p \in U$, then there exists a $\mathscr{T}_1$-open set V (and thus $\mathscr{T}$-open) such that $p \in V \subset cl^*(V) \subset cl^1(V) \subset U \subset \cup\{U_\beta \cap B_\beta | \beta \in \Gamma\} \cup U$ since $(X, \mathscr{T}_1)$ is regular. Now suppose $p \notin U$. Then $p \in (U_\beta \cap B_\beta)$ for some $\beta \in \Gamma$. Since $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ are regular, there exist $\mathscr{T}_1$-open and $\mathscr{T}_2$-open sets V_β and D_β respectively such that $p\in V_\beta\subset \mathrm{cl}^1(V_\beta)\subset U_\beta$ and $p\in D_\beta\subset \mathrm{cl}^2(D_\beta)\subset B_\beta$. Now $X-\mathrm{cl}^1(V_\beta)$ is $\mathscr{T}_1$-open (and thus $\mathscr{T}$-open), and $X-\mathrm{cl}^2(D_\beta)$ is $\mathscr{T}_2$-open (and thus $\mathscr{T}$-open). Consequently $[X-\mathrm{cl}^1(V_\beta)]\cup[X-\mathrm{cl}^2(D_\beta)]=X-[\mathrm{cl}^1(V_\beta)\cap \mathrm{cl}^2(D_\beta)]$ is $\mathscr{T}$-open so that $\mathrm{cl}^1(V_\beta)\cap \mathrm{cl}^2(D_\beta)$ is $\mathscr{T}$-closed. Therefore we have $p\in(V_\beta\cap D_\beta)\subset \mathrm{cl}^*(V_\beta\cap D_\beta)\subset \mathrm{cl}^1(V_\beta)\cap \mathrm{cl}^2(D_\beta)\subset(U_\beta\cap B_\beta)$ since $\mathrm{cl}^*(V_\beta\cap D_\beta)$ is the smallest $\mathscr{T}$-closed set containing $V_\beta\cap D_\beta$. Obviously $(V_\beta\cap D_\beta)$ is $\mathscr{T}$-open. Hence $(X, \mathscr{T})$ is regular.

Before proving the next theorem we shall need the following lemma.

LEMMA 10.1. *An arbitrary $\mathscr{T}$-closed subset of X is of the form $\cap\{K_\beta\cup(G_\beta\cap F)\mid\beta\in\Gamma\}$ where F, K_β $(\beta\in\Gamma)$ are $\mathscr{T}_1$-closed subsets of X and the G_β $(\beta\in\Gamma)$ are $\mathscr{T}_2$-closed subsets of X.*

PROOF. Let M be an arbitrary $\mathscr{T}$-closed subset of X. Then there exists a $\mathscr{T}$-open set $\cup\{U_\beta\cap B_\beta\mid\beta\in\Gamma\}\cup U$ (where $U_\beta, U\in\mathscr{T}_1$, $\beta\in\Gamma$) such that $M=X-[\cup\{U_\beta\cap B_\beta\mid\beta\in\Gamma\}\cup U]=\cap\{X-(U_\beta\cap B_\beta)\mid\beta\in\Gamma\}\cap(X-U)=\cap\{(X-U_\beta)\cup(X-B_\beta)\mid\beta\in\Gamma\}\cap(X-U)=\cap\{F_\beta\cup G_\beta\mid\beta\in\Gamma\}\cap F$ where $F=(X-U)$, $F_\beta=(X-U_\beta)$, $\beta\in\Gamma$, are $\mathscr{T}_1$-closed and the $G_\beta=(X-B_\beta), \beta\in\Gamma$, are $\mathscr{T}_2$-closed. Now we can write $\cap\{F_\beta\cup G_\beta\mid\beta\in\Gamma\}\cap F=\cap\{K_\beta\cup(G_\beta\cap F)\mid\beta\in\Gamma\}$ where $K_\beta=(F_\beta\cap F)$, $\beta\in\Gamma$.

THEOREM 10. *Let $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ be completely regular spaces. Then $(X, \mathscr{T})$ is completely regular.*

PROOF. Let p be an arbitrary point in X and let $\cap\{K_\beta\cup(G_\beta\cap F)\mid\beta\in\Gamma\}$ be an arbitrary $\mathscr{T}$-closed set (where K_β, F are $\mathscr{T}_1$-closed, $\beta\in\Gamma$, and $G_\beta=X-B_\beta$, $\beta\in\Gamma$) such that $p\notin\cap\{K_\beta\cup(G_\beta\cap F)\mid\beta\in\Gamma\}$. Then there exists at least one $\beta'\in\Gamma$ such that $p\notin[K_{\beta'}\cup(G_{\beta'}\cap F)]$ and this implies that $p\notin K_{\beta'}$ and $p\notin(G_{\beta'}\cap F)$. And $p\notin(G_{\beta'}\cap F)$ implies that $p\notin G_{\beta'}$ or $p\notin F$.

Case 1. $p\notin K_{\beta'}$ and $p\notin G_{\beta'}$. Then there exist real-valued functions f ($\mathscr{T}_1$-continuous) and g ($\mathscr{T}_2$-continuous) on X such that $f(p)=g(p)=0$, $f(K_{\beta'})=g(G_{\beta'})=1$, and $0\leq f(x)\leq 1$, $0\leq g(x)\leq 1$ for every $x\in X$ since $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ are completely regular. Set $h(x)=\max[f(x), g(x)]$, $x\in X$. We shall show that h is a $\mathscr{T}$-continuous function from X onto $\mathscr{I}=[0, 1]$. Let O be an arbitrary $\mathscr{U}\cap\mathscr{I}$-open set ($\mathscr{U}\cap\mathscr{I}$ is the relative usual topology on $\mathscr{I}$). Then $O=\cup\{I_\alpha\mid\alpha\in\Lambda\}$ where the I_α are $\mathscr{U}\cap\mathscr{I}$-open intervals. Now $h^{-1}(O)=h^{-1}(\cup\{I_\alpha\mid\alpha\in\Lambda\})=\cup\{h^{-1}(I_\alpha)\mid\alpha\in\Lambda\}$. Suppose I_α has right endpoint b_α. Since $h(x)=\max[f(x), g(x)]$, $x\in X$ we have $h^{-1}(I_\alpha)=\{f^{-1}(I_\alpha)-g^{-1}([b_\alpha, 1])\}\cup\{g^{-1}(I_\alpha)-f^{-1}([b_\alpha,$

1]}. (Note: if $b_\alpha=1$ and $b_\alpha\in I_\alpha$ we shall set $[b_\alpha, 1]=\phi$; if $b_\alpha=1$ and $b_\alpha\notin I_\alpha$ then let $[b_\alpha, 1]=\{1\}$.) Since g is $\mathscr{T}_2$-continuous it is also $\mathscr{T}$-continuous because $\mathscr{T}_2\subset\mathscr{T}$. Hence $g^{-1}([b_\alpha, 1])$ is a $\mathscr{T}$-closed set. Similarly since f is $\mathscr{T}_1$-continuous, it is also $\mathscr{T}$-continuous and so $f^{-1}(I_\alpha)$ is $\mathscr{T}$-open. Therefore $\{f^{-1}(I_\alpha)-g^{-1}([b_\alpha, 1])\}$ is a $\mathscr{T}$-open set. It can be shown, in a similar manner, that $\{g^{-1}(I_\alpha)-f^{-1}([b_\alpha, 1])\}$ is a $\mathscr{T}$-open set. Consequently $h^{-1}(O)$ is a $\mathscr{T}$-open set so that h is a $\mathscr{T}$-continuous function from X onto $\mathscr{I}$.

Case 2. $p\notin K_{\beta'}$ and $p\notin F$. Then there exist real-valued functions $f(\mathscr{T}_1$-continuous) and $g(\mathscr{T}_1$-continuous) on X such that $f(p)=g(p)=0$, $f(K_{\beta'})=g(F)=1$, and $0\leq f(x)\leq 1$, $0\leq g(x)\leq 1$ for every $x\in X$ since $(X, \mathscr{T}_1)$ is completely regular. Set $h(x)=\max[f(x), g(x)]$, $x\in X$. The proof that h is $\mathscr{T}$-continuous from X onto $\mathscr{I}=[0, 1]$ is similar to that in Case 1.

In either case above we have $h(p)=0$ and $h(\bigcap\{K_\beta\cup(G_\beta\cap F)\mid\beta\in\Gamma\})=1$ since $\bigcap\{K_\beta\cup(G_\beta\cap F)\mid\beta\in\Gamma\}\subset[K_{\beta'}\cup(G_{\beta'}\cap F)]$ and $h[K_{\beta'}\cup(G_{\beta'}\cap F)]=1$. Thus $(X, \mathscr{T})$ is completely regular.

Now if $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ are both normal, the space $(X, \mathscr{T})$ is not necessarily normal, as is shown in the following example.

EXAMPLE 1. Let S be the set of non-negative real numbers and let $\mathscr{R}$ be the topology on S whose basis is made up of right-half open intervals; i.e., intervals of the form $\{x\mid a\leq x<b\}$ where $0\leq a<b$. The topological space $(S, \mathscr{R})$ is Hausdorff and is shown to be paracompact by Sorgenfrey [3]. The space $(S, \mathscr{U})$, where $\mathscr{U}$ is the usual topology on S whose basis is made up of open intervals (i.e., intervals of the form $\{x\mid a<x<b\}$ where $0\leq a<b$ and intervals of the form $[0, b)$ for $\mathscr{U}$-neighborhoods of 0), is countable at infinity (see Kowalsky [1, p. 90]). Let $X=S\times S$, $\mathscr{T}_1=\{U\times V\mid U\in\mathscr{R}, V\in\mathscr{U}\}$, and $\mathscr{T}_2=\{U'\times V'\mid U'\in\mathscr{U}, V'\in\mathscr{R}\}$. Consider the topological space $(X, \mathscr{T}_1)$. By a theorem from Kowalsky [1, p. 153] $(X, \mathscr{T}_1)$ is paracompact. It is obviously Hausdorff. Thus $(X, \mathscr{T}_1)$ is normal. Similarly $(X, \mathscr{T}_2)$ is normal. Let $G_2=\{B_\beta\mid B_\beta$ is a member of an open basis for $\mathscr{T}_2$ such that $B_\beta\notin\mathscr{T}_1$, $\beta\in\Gamma\}$. The B_β are of the form $(a, b)\times[c, d)$ where a, b, c, d are non-negative real numbers. Now consider the space $(X, \mathscr{T})=(X, \mathscr{T}_1[B_\beta])$. The topology $\mathscr{T}$ contains sets of the form $[a, b)\times[c, d)$ where a, b, c, d are nonnegative real numbers. Let H be the set of all points (x, y) of X such that $x+y=1$ and $[(x-1)^2+y^2]^{1/2}$ is rational, and let K be the set of all points (x, y) of X such

that $x+y=1$ and $[(x-1)^2+y^2]^{1/2}$ is irrational. The sets H and K are disjoint $\mathscr{T}$-closed subsets of X which cannot be separated by $\mathscr{T}$-open sets (see Sorgenfrey [3, p. 632] for the proof that $(S\times S,\ \mathscr{R}\times\mathscr{R})$ is not normal). Thus $(X,\ \mathscr{T})$ is non-normal.

In conclusion we give an example in which $(X,\ \mathscr{T})$ is non-compact, and $(X,\ \mathscr{T}_1)$ and $(X,\ \mathscr{T}_2)$ are both compact spaces.

EXAMPLE 2. Let $X=\{(x,\ y)\,|\,0<x\leq 1,\ 0\leq y\leq 1\}$; i.e., X is a closed unit square with its left boundary removed. Consider the topology $\mathscr{T}_1$ on X which consists of usual open sets for a Möbius band represented by X. A $\mathscr{T}_1$-neighborhood basis for a point $p=(x,\ y)$ where $0<x<1$, $0<y<1$ is just the collection of open spheres about p contained in X. A $\mathscr{T}_1$-neighborhood basis for a point p on the upper or lower boundary of X (which is not on the right boundary of X) is just the collection of open hemispheres about p contained in X. For a point $p=(x,\ y)$ on the right boundary of X, say $x=1$, $y=y_1$, a member of the $\mathscr{T}_1$-basis for p would be an open hemisphere (intersected with X) about p, together with the open hemisphere (intersected with X) about $(0, 1-y_1)$ of the same radius as the hemisphere about p. Let us denote such a member of $\mathscr{T}_1$ by $O_1\,(y_1,\ r)=W(y_2,\ r)\cup V(y_1,\ r)$ where $W(y_1,\ r)$ is the hemisphere about $(1,\ y_1)$, $V(y_1,\ r)$ is the hemisphere about $(0,\ 1-y_1)$, and r is the radius of the hemispheres. It is obvious that $(X,\ \mathscr{T}_1)$ is compact. Now consider the topology $\mathscr{T}_2$ on X which consists of usual open sets for a right circular cylinder cut along one of its generators and folded out (represented by X). The $\mathscr{T}_2$-neighborhood bases of all points in X except those on the right boundary are the same as the $\mathscr{T}_1$-neighborhood bases. For a point $p=(1,\ y_1)$ on the right boundary of X, a member of the $\mathscr{T}_2$-basis for p would be an open hemisphere (intersected with X) about p, together with the open hemisphere (intersected with X) about $(0,\ y_1)$ of the same radius as the hemisphere about p. Let us denote such a member of $\mathscr{T}_2$ by $O_2(y_1,\ r)=W(y_1,\ r)\cup U(y_1,\ r)$ where $W(y_1,\ r)$ is defined as above, $U(y_1,\ r)$ is the hemisphere about $(0,\ y_1)$, and r is the radius of the hemispheres. It is also obvious that $(X,\ \mathscr{T}_2)$ is compact.

Let $G_2=\{B_\beta\,|\,B_\beta$ is a member of an open basis for $\mathscr{T}_2$ such that $B_\beta\notin\mathscr{T}_1$, $\beta\in\Gamma\}$. Thus G_2 is the collection of $\mathscr{T}_2$-neighborhood bases for all points on the right boundary of X. Now if $y_1\neq 1/2$ and $r<|1/2-y_1|$ we have $O_1(y_1,\ r)$

$\cap O_2(y_1, r) = W(y_1, r)$. Hence $W(y_1, r) \in \mathscr{T} = \mathscr{T}_1[B_\beta]$ if $r < |1/2 - y_1|$ and $y_1 \neq 1/2$. Now consider the following $\mathscr{T}$-open covering of X:

$$\mathscr{C} = \{W(y_1, r) \mid y_1 \neq 1/2 \text{ and } r < |1/2 - y_1|\} \cup \{O_1(1/2, 1/4)\}$$
$$\cup \{U_n \mid U_n = \{(x, y) \mid 1/n < x < 1, 0 \leq y \leq 1\}, n = 2, 3, \cdots\}.$$

The covering $\mathscr{C}$ obviously has no finite subcovering. Hence $(X, \mathscr{T})$ is non-compact.

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