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ANALYTIC SUFFICIENCY CONDITIONS FOR GOLDBACH'S CONJECTURE

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1. Introduction

It is generally known among serious students of the Goldbach conjecture that within the framework of the Hardy-Littlewood circle method attack on the problem the whole difficulty is in obtaining the requisite estimate on the integral of the representation function over the minor arcs.

The purpose of this paper is to fill a gap in the literature by carefully elaborating upon the above statement; as we have not been able to find any of our results even mentioned in print.

2. The Hardy-Littlewood circle method

In this section we will present an outline of the Hardy-Littlewood circle method as modified by Vinogradov in [77] and as presented by Estermann in [13].

Let $\varepsilon(\alpha)=\varepsilon^{2\pi i\alpha}$, $f(x, v)=\sum_{p\leq v}\varepsilon(px)$, $x_0=\frac{\log^{15}n}{n}$

For each n we will select a finite number of rational points in the closed interval $[x_0, x_0+1]$. Symmetrically placed about each of these rational points will be a closed neighborhood of radius x_0. Each such closed neighborhood will be called a *major arc,* and the union of all such major arcs for each fixed n will be denoted, $M(n)$. Everything will be arranged so that for each fixed n (greater than some fixed integer N_0) the major arcs will be pairwise disjoint. For each fixed n each interval between adjacent major arcs (or between a major arc and the point x_0) will be called a *minor arc,* and the union of all such minor arcs will be denoted $m(n)$. Clearly, for each n, $m(n)\cap M(n)=\Phi$ and $m(n)\cup M(n)=[x_0, x_0+1]$.

The crux of our application of the Hardy-Littlewood circle method will be the construction of three sequences $R(n)$, $S(n)$, and $T(n)$ with the following properties:

1. $\frac{1}{3}n\log^{-2}n\leq T(n)$.

2. $\int_{M(n)} f^2(x, n)\varepsilon(-nx)dx = T(n)R(n) + O(n\log^{-A}n)$.

3. $S(n) - R(n) = o(1)$.

4. There exists a real number, θ, such that $S(n) \geq \theta > 0$ if n is an even integer.

We conclude this section with the following.

LEMMA 2.1. *Let* $\{A_n\}$ *be any sequence of subsets of* $[x_0, x_0+1]$. *Then*

$$\int_{A_n} f^2(x, n)\varepsilon(-nx)dx = O(n\log^{-1}n).$$

PROOF.

$$\left|\int_{A_n} f^2(x, n)\varepsilon(-nx)dx\right| \leq \int_{x_0}^{x_0+1} \left|f^2(x, n)\right|dx \leq \int_0^1 \left|f(x, n)\right|^2 dx = \Pi(n).$$

3. An analytic sufficiency condition with the generalized Riemann hypothesis

At the very end of his expository talk to the Mathematical Society of Copenhagen in 1921 (cf. [24]) Hardy made under the assumption of Hypothesis R the following statement after discussing the case $r=3$ which corresponds to his Theoerm D in [25].

There is unfortunately a vital difference between the case $r=2$, which corresponds to Goldbach's theorem, and all of the rest. We have to fill in the skeleton which I have presented to you, and to transform it into an accurate proof; and in doing this we find ourselves compelled to suppose that $r>2$. It only remains that I should explain to you shortly the reason for this regrettable limitation. The explanation which follows must be taken merely as a first approximation to the truth.

We will not elaborate further upon this statement by Hardy other than to say that it is clear that Hardy and Littlewood in 1921 knew in addition to their assumption of Hypothesis R precisely what the difficulty was with regard to their method of attack on the Goldbach conjecture.

The rest of this section will be devoted to exhibiting a plausible analytic sufficiency condition for the asymptotic formulation of Goldbach's conjecture under the assumption of the generalized Riemann hypothesis by means of Estermann's formulation of Vinogradov's modification of the Hardy-Littlewood circle method.

First we present some preliminary lemmata and theorems, some of which will be used in Section 4.

LEMMA 3.1. *For any numbers x_1 and x_2,*

$$f(x_1+x_2,\ v)=\varepsilon(vx_2)f(x_1,\ v)-2\pi ix_2\int_0^v \varepsilon(\mu x_2)f(x_1,\ \mu)du.$$

PROOF. [13] page 53.

Let $g(x,\ v)=\sum_{2\leq m\leq v}\frac{\varepsilon(mx)}{\log m}$ $(v\geq 2)$ and $g(x, v)=0$ $(v<2)$.

LEMMA 3.2. *For any numbers x_1 and x_2,*

$$g(x_1+x_2,\ v)=\varepsilon(vx_2)g(x_1,\ v)-2\pi ix_2\int_0^v \varepsilon(\mu x_2)g(x_1,\ \mu)d\mu.$$

PROOF. [13] page 63.

LEMMA 3.3. $C_n(m)=\sum_{d/m,\, d/n}\mu\left(\frac{n}{d}\right)d.$

PROOF. [28] page 237.

THEOREM 3.1. *Let $m\geq 3$, $k\leq m^{1/2}(\log m)^A$* and *$(k,\ l)=1$. Then under the assumption of the generalized Riemann hypothesis for every $\delta>0$*

$$\left|\Pi(m:k,\ l)-\frac{l\ sm}{\phi(k)}\right|\leq A(\delta)m^{\frac{1}{2}+\delta}.$$

PROOF. [9] page 129.

THEOREM 3.2. *Let $m\geq 3$, $k\leq \log^{\mu} m$ and $(k,\ l)=1$. Then*

$$\left|\Pi(m:\ k,\ l)-\frac{l\ sm}{\phi(k)}\right|\leq Am\ exp\left(-\frac{\sqrt{\log m}}{200}\right).$$

PROOF. [13] Chapter 2.

LEMMA 3.4.

$$\sum_{n\leq x}\frac{1}{\phi(n)}=\frac{\xi(2)\xi(3)}{\xi(6)}\log x+A+O\left(\frac{\log x}{x}\right).$$

PROOF. [46] page 38.

LEMMA 3.5.

$$\sum_{q=1}^{\infty}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)-\sum_{q\leq y}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)=O(d(n)(\log\log 3y)^2y^{-1})$$

PROOF. [49] page 211.

LEMMA 3.6. *$d(n)=O(n^{\varepsilon})$ for enery $\varepsilon>0$.*

PROOF. [28] page 260.

LEMMA 3.7. *$\frac{\phi(n)}{n^{1-\varepsilon}}\to\infty$ for every $\varepsilon>0$.*

PROOF. [28] page 267.

LEMMA 3.8. *If $f(q)$ is multiplicative, and $\sum_{n=1}^{\infty}|f(q)|<\infty$, then*

$$\sum_{q=1}^{\infty}f(q)=\prod_{p}\sum_{m=0}^{\infty}f(p^m).$$

PROOF. [13] page 3.

LEMMA 3.9. *If $(n, q)=a$ and $q=aN$, then*

$$C_q(n)=\sum_{\substack{0<h\leq q\\(h,q)=1}}\varepsilon\left(-\frac{nh}{q}\right)=\frac{\mu(N)\phi(q)}{\phi(N)}.$$

PROOF. [28] page 238.

LEMMA 3.10. *Corresponding to any x and any $y\geq 1$ there are numbers h, q such that $(h, q)=1$ and such that $q\leq y$ and $|qx-h|<y^{-1}$.*

PROOF. [28] page 30.

Restrict ε such that $0<\varepsilon<\frac{1}{100}$. For each n let $m(n)$ be those points in $[x_0, x_0+1]$ which are not in any closed neighborhood (major arc) of radius x_0 about any rational number $\frac{h}{q}$ where $(h, q)=1$ and $q\leq n^{\varepsilon}$. If $n\geq N_0$, then the major arcs are pairwise disjoint, since

$$\left(\frac{h_2}{q_2}-\frac{h_1}{q_1}\right)=\frac{h_2q_1-h_1q_2}{q_1q_2}\geq\frac{1}{q_1q_2}\geq\frac{1}{n^{2\varepsilon}}>2x_0$$

$$=\frac{2\log^{15}n}{n},\ \textit{if } n\geq N_0.$$

Clearly, the measure of $M(n)$ is less than or equal to $\left(\frac{2\log^{15}n}{n^{1-2\varepsilon}}\right)$; so that since $\varepsilon<\frac{1}{2}$ it tends to zero; so that the measure of $m(n)$ tends to one.

Let $r(n)$ be the number of representations of n as the sum of two primes. It is easy to see that

$$r(n)=\int_{x_0}^{x_0+1} f^2(x,n)\varepsilon(-nx)dx \text{ for any } x_0.$$

We decompose the above integral into

$$r(n)=\int_{m(n)} f^2(x, n)\varepsilon(-nx)dx+\int_{M(n)} f^2(x, n)\varepsilon(-nx)dx$$
$$=A(n)+B(n)$$

By Lemma 2.1 we know that $A(n)=O(n\log^{-1}n)$.

We now establish the following.

THEOREM 3.3. *Assume the generalized Riemann hypothesis. Then* $A(n)=o(n\log^{-2}n)$ *implies* $r(n)>0$ *for every even* $n\geq N_0$.

PROOF. By definition

$$\int_{M(n)} f^2(x, n)\varepsilon(-nx)dx=\sum_{q\leq n^{\varepsilon}}\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h, q),$$

where

$$T(h, q)=\int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} f^2(x, n)\varepsilon(-nx)dx.$$

LEMMA 3.11. *Let* $q\leq n^{\varepsilon}$, $|y|\leq x_0$, $(h, q)=1$, *and* $n\geq N_0$. *Then*

$$\left|f\left(\frac{h}{q}+y, n\right)-\frac{\mu(q)}{\phi(q)}g(y, n)\right|\leq C_2(\log^{15}n)n^{1/2+\varepsilon+\delta}.$$

PROOF. By the definition of $f(x, v)$ we have

$$\left|f\left(\frac{h}{q}, v\right)-\sum_{\substack{p\leq v\\p\nmid q}}\varepsilon\left(\frac{ph}{q}\right)\right|\leq\sum_{p|q}1<q.$$

But it is easy to see

$$\sum_{\substack{p\le v\\ p\nmid q}} \varepsilon\Big(\frac{ph}{q}\Big)=\sum_{\substack{0<l\le q\\ (l,q)=1}}\Big(\varepsilon\Big(\frac{lh}{q}\Big)\sum_{\substack{p\le v\\ p\equiv l (\mathrm{mod}\, q)}} 1\Big)$$

$$=\sum_{\substack{0<l\le q\\ (l,q)=1}}\Big(\varepsilon\Big(\frac{lh}{q}\Big)\Pi([v]\ ;q,\ l)\Big),$$

and by Theorem 3.1 we have for some $\delta<\frac{1}{100}$

$$\Big|\Pi([v]\ ;q,\ l)-\frac{ls[v]}{\phi(q)}\Big|\le n^{1/2+\delta}(0\le v\le n,\ (q,\ l)=1 \text{ and } n\ge N_0).$$

But by definition $ls[v]=q(0,\ v)$, and since $(h,\ q)=1$ we have by Lemma 3.3 that

$$\sum_{\substack{0<l\le q\\ (l,q)=1}}\varepsilon\Big(\frac{lh}{q}\Big)=\mu(q).$$

Hence for $0\le v\le n$ and $n\ge N_0$,

$$\Big|f\Big(\frac{h}{q},\ v\Big)-\frac{\mu(q)}{\phi(q)}g(0,\ v)\Big|<q+\Big|\sum_{\substack{p\le v\\ p\nmid q}}\varepsilon\Big(\frac{ph}{q}\Big)-\frac{\mu(q)}{\phi(q)}ls[v]\Big|$$

$$=q+\Big|\sum_{\substack{0<l\le q\\ (l,q)=1}}\varepsilon\Big(\frac{lh}{q}\Big)\Big\{\Pi([v]\ ;q,\ l)-\frac{ls[v]}{\phi(q)}\Big\}\Big|$$

$$\le q+qn^{1/2+\delta}\le n^{\varepsilon}+n^{1/2+\varepsilon+\delta}\le C_1n^{1/2+\varepsilon+\delta}\ (0\le v\le n).$$

By Lemma 3.1 and Lemma 3.2 we have

$$f\Big(\frac{h}{q}+y,\ n\Big)=\varepsilon(ny)f\Big(\frac{h}{q},\ v\Big)-2\pi iy\int_0^n\varepsilon(vy)f\Big(\frac{h}{q},\ v\Big)dv,$$

and

$$g(y,\ n)=\varepsilon(ny)g(0,\ n)-2\pi iy\int_0^n\varepsilon(vy)g(0,\ v)dv.$$

Hence, using the fact that $|y|\le x_0$ and $x_0=\frac{\log^{15}n}{n}$ we have

$$\Big|f\Big(\frac{h}{q}+y,\ n\Big)-\frac{\mu(q)}{\phi(q)}g(y,\ n)\Big|$$

$$=\Big|\varepsilon(ny)\Big\{f\Big(\frac{h}{q},\ n\Big)-\frac{\mu(q)}{\phi(q)}g(0,\ n)\Big\}-2\pi iy\int_0^n\varepsilon(vy)\Big\{f\Big(\frac{h}{q},\ v\Big)-\frac{\mu(q)}{\phi(q)}g(0,\ v)\Big\}dv\Big|$$

$$\le\Big|f\Big(\frac{h}{q},\ n\Big)-\frac{\mu(q)}{\phi(q)}g(0,\ n)\Big|+2\pi x_0\int_0^n\Big|f\Big(\frac{h}{q},\ v\Big)-\frac{\mu(q)}{\phi(q)}g(0,\ v)\Big|dv$$

$$\leq (1+2\pi x_0 n) C_1 n^{(1/2+\varepsilon+\delta)}$$

$$\leq C_2 (\log^{15} n) n^{1/2+\varepsilon+\delta} \text{ for } n \geq N_0.$$

LEMMA 3.12. *Under the hypothesis of Lemma* 3.11 *we have*

$$\left| f^2\left(\frac{h}{q}+y,\ n\right) - \frac{\mu^2(q)}{\phi^2(q)} g^2(y,\ n) \right| \leq C_3 (\log^{15} n) n^{3/2+\varepsilon+\delta}.$$

PROOF. This is immediate by Lemma 3.11 and the trivial inequalities $|f(x, n)| \leq n$ and $|g(y, n)| \leq n$ and the fact that if $|a| \leq n$ and $|b| \leq n$, then $|a^2-b^2| \leq 2n|a-b|$.

We now assume $n \geq N_0$ throughout the rest of the proof.

By a change of variable $y=\left(x-\frac{h}{q}\right)$ we have

$$T(h,\ q) = \varepsilon\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y,\ n\right) \varepsilon(-ny) dy \tag{A}$$

However, by Lemma 3.12

$$\left| \varepsilon\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y,\ n\right) \varepsilon(-ny) dy - \frac{\mu^2(q)}{\phi^2(q)} \varepsilon\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} g^2(y,\ n) \varepsilon(-ny) dy \right|$$

$$\leq \int_{-x_0}^{x_0} \left| f^2\left(\frac{h}{q}+y,\ n\right) - \frac{\mu^2(q)}{\phi^2(q)} g^2(y,\ n) \right| dy \leq \int_{-x_0}^{x_0} C_3 (\log^{15} n) n^{(3/2+\varepsilon+\delta)} dy$$

$$\leq 2C_3 x_0 (\log^{15} n) n^{3/2+\varepsilon+\delta} \leq C_4 (\log^{30} n) n^{1/2+\varepsilon+\delta}.$$

Now let $T_1(n) = \int_{-x_0}^{x_0} g^2(y,\ n) \varepsilon(-ny) dy$; so that by (A) and the above we have if $(h,\ q)=1$ and $q \leq n^{\varepsilon}$, then

$$\left| T(h, q) - \frac{\mu^2(q)}{\phi^2(q)} T_1(n) \varepsilon\left(-\frac{nh}{q}\right) \right| \leq C_4 (\log^{30} n)(n^{1/2+\varepsilon+\delta}) \tag{B}$$

$$\text{Let } T(n) = \sum_{m_1,\, m_2} \log^{-1} m_1 \log^{-1} m_2 \tag{C}$$

with the conditions of summation $m_1 \geq 2$, $m_2 \geq 2$, and $m_1+m_2=n$.

It is easy to see that $T(n) = \int_{-1/2}^{1/2} g^2(y,\ n) \varepsilon(-ny) dy.$ (D)

Also, it is clear that the number of terms on the righthand side of (C) is

$(n-3)$, and each term is greater than $\log^{-2}n$ and less than 1: so that $\frac{1}{3}n \log^{-2}n < T(n)$. (E)

It is easy to see using the formula for the sum of a geometric series that

$$\left|\sum_{m=2}^{m_1} \varepsilon(my)\right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2|y|} \ ; (m_1 \geq 2,\ 0<|y|\leq \tfrac{1}{2}).$$

Hence by the definition of $g(y, n)$ and Abel's lemma,

$$|g(y,\ n)| \leq |y|^{-1} \qquad \left(0<|y|\leq \tfrac{1}{2}\right);$$

so that

$$|T(n)-T_1(n)| \leq 2\int_{x_0}^{1/2} y^{-2}dy < 2x_0^{-1} = 2n\log^{-15}n. \tag{F}$$

Hence for $(h,\ q)=1,\ q\leq n^{\varepsilon}$

$$\left|\varepsilon\left(-\frac{nh}{q}\right)\right| \left|\frac{\mu^2(q)}{\phi^2(q)}\right| \left|T(n)-T_1(n)\right| \leq \frac{1}{\phi^2(q)}(2n\log^{-15}n),$$

and combining this fact with (B) we have

$$\left|T(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)\varepsilon\left(-\frac{nh}{q}\right)\right| \leq C_4(\log^{30}n)(n^{1/2+\varepsilon+\delta}) + \frac{1}{\phi^2(q)}(2n\ \log^{-15}n)\ ; \tag{G}$$

so that that adding (G) $\phi(q)$ times for some fixed $q\leq n^{\varepsilon}$ we have

$$\left|\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h,\ q)-\frac{u^2(q)}{\phi^2(q)}T(n)\sum_{\substack{0<h\leq q\\(h,q)=1}}\varepsilon\left(-\frac{nh}{q}\right)\right|$$
$$\leq C_4(\log^{30}n)(n^{1/2+\varepsilon+\delta})\phi(q)+\frac{1}{\phi(q)}(2n\ \log^{-15}n). \tag{H}$$

But $\phi(q)\leq n^{\varepsilon}$ and by definition

$$\sum_{\substack{0<h\leq q\\(h,q)=1}}\varepsilon\left(-\frac{nh}{q}\right)=C_q(n)\ ;$$

so that it follows immediately from (H) that:

$$\left|\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)C_q(n)\right|$$
$$\leq C_4(\log^{30}n)(n^{1/2+\varepsilon+\delta})+\frac{1}{\phi(q)}(2n\ \log^{-15}n). \tag{I}$$

Now summing over all $q\leq n^{\varepsilon}$ and using Lemma 3.4 we have

$$\left|\sum_{q\le n^\varepsilon}\sum_{\substack{0<h\le q\\(h,q)=1}} T(h,\ q)-T(n)\sum_{q\le n^\varepsilon}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\right|$$

$$\le C_4(\log^{30} n)n^{1/2+3\varepsilon+\delta}+\Big[\sum_{q\le n^\varepsilon}\frac{1}{\phi(q)}\Big](2n\ \log^{-15} n)$$

$$\le C_4(\log^{30} n)n^{1/2+3\varepsilon+\delta}+\Big(C_5\ \log n^\varepsilon+C_6+C_7\frac{\log n^\varepsilon}{n^\varepsilon}\Big)(2n\ \log^{-15} n)$$

$$\le C_4(\log^{30} n)n^{1/2+3\varepsilon+\delta}+C_8 n\ \log^{-14} n+C_9 n\ \log^{-15} n$$

$$+C_{10}n^{1-\varepsilon}\log^{-14} n$$

$$\le C_{11}n\ \log^{-14} n\ ;$$

Hence this estimate with the hypothesis yields

$$\left|r(n)-T(n)\sum_{q\le n^\varepsilon}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\right|\le k(n)n\ \log^{-2} n+C_{11}n\ \log^{-14} n\ ; \qquad \text{(J)}$$

where $k(n)\to 0$. Now let

$$R(n)=\sum_{q\le n^\varepsilon}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\ \textit{and}\ S(n)=\sum_{q=1}^{\infty}\frac{\mu^2(q)}{\phi^2(q)}C_q(n).$$

By Lemma 3.5 and Lemma 3.6 we have that

$$|S(n)-R(n)|\le C_{12}\frac{d(n)(\log\ \log 3n^\varepsilon)^2}{n^\varepsilon}\le C_{13}\frac{n^{\frac{\varepsilon}{2}}(\log\ \log 3n^\varepsilon)^2}{n^\varepsilon}\ ;$$

so that $|S(n)-R(n)|=0(1)$.

By (E) what remains to be done is to show that $S(n)$ is uniformly bounded away from zero.

Let $f(q)=\dfrac{\mu^2(q)}{\phi^2(q)}C_q(n)$.

Since $\mu(q)$, $\phi(q)$ and $C_q(n)$ are all multiplicative functions of q, f is a multiplicative function of q. Also, by means of the trivial estimate on $C_q(n)$, namely n, and a direct application of Lemma 3.7 we have

$$\sum_{q=1}^{\infty}|f(q)|\le n\sum_{q=1}^{\infty}\frac{1}{\phi^2(q)}<\infty \text{ for each } n\ ;$$

so that by Lemma 3.8 we have for each n

$$S(n)=\prod_q\sum_{m=0}^{\infty}f(p^m).$$

But

$$If\ m=0,\ f(p^m)=f(p^0)=f(1)=\frac{\mu^2(1)}{\phi^2(1)}C_1(n)=1$$

$$If\ m=1,\ f(p^m)=f(p^1)=f(p)=\frac{\mu^2(p)}{\phi^2(p)}C_p(n)=\frac{C_p(n)}{(p-1)^2}$$

$$If\ m\geq 2,\ \mu(p^m)=0\ ;\ so\ that\ f(p^m)=0.$$

Hence

$$S(n)=\prod_p\left(1+\frac{C_p(n)}{(p-1)^2}\right).$$

But by Lemma 3.9 we have $C_p(n)=(p-1)$ if $(p,\ n)>1$, and $C_p(n)=-1$ if $(p,\ n)=1$; so that

$$S(n)=2\prod_{p>2}\left(1+\frac{C_p(n)}{(p-1)^2}\right)\geq 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\geq 2\prod_{m=2}^{\infty}\left(1-\frac{1}{m^2}\right)=1.$$

4. An analytic sufficiency condition without the generalized Riemann hypothesis

One, of course, would prefer an analytic sufficiency condition for the asymptotic formulation of Goldbach's conjecture which would not require the generalized Riemann hypothesis. Mozzochi (c. 1975) in [53] established such a condition, and we will improve his results here and in Section 5.

For each n let $m(n)$ be those points in $[x_0,\ x_0+1]$ which are not in any closed neighborhood (major arc) of radius x_0 about any rational number $\frac{h}{q}$ where $(h,\ q)=1$, $(q,\ n)=1$ and $q\leq\log^{15}n$. If $n\geq N_0$, then the major arcs are pairwise disjoint. Clearly, the measure of $M(n)$ is less than or equal to $\left(\frac{2\log^{45}n}{n}\right)$; so that it tends to zero. Consequently, the measure of $m(n)$ tends to one.

Let $r(n)$ be the number of representations of n as the sum of two primes. It is easy to see that

$$r(n)=\int_{x_0}^{x_0+1} f^2(x,\ n)\varepsilon(-nx)dx\ for\ any\ x_0.$$

We decompose the above integral into

$$r(n)=\int_{m(n)} f^2(x,\ n)\varepsilon(-nx)dx+\int_{M(n)} f^2(x,\ n)\varepsilon(-nx)dx$$

$=A(n)+B(n).$

By Lemma 2.1 we know that $A(n)=O(n \log^{-1} n)$.

We now establish the following.

THEOREM 4.1. *$A(n)=o(n \log^{-2} n)$ implies $r(n)>0$ for every even $n \geq N_0$.*

PROOF. By definition

$$\int_{m(n)} f^2(x, n)\varepsilon(-nx)dx = \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \sum_{\substack{0<h\leq q \\ (h,q)=1}} T(h, q),$$

where

$$T(h, q) = \int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} f^2(x, n)\varepsilon(-nx)dx.$$

LEMMA 4.1. *Let $q \leq \log^{15} n$, $|y| \leq x_0$, $(h, q)=1$, and $n \geq N_0$. Then*

$$\left| f\left(\frac{h}{q}+y, n\right) - \frac{\mu(q)}{\phi(q)} g(y, n) \right| \leq n \log^{-69} n.$$

PROOF. This follows from Theorem 3.2 in a way very similar to the way that Lemma 3.11 follows from Theorem 3.1. See Theorem 58 in [13].

LEMMA 4.2. *Under the hypothesis of Lemma 4.1 we have*

$$\left| f^2\left(\frac{h}{q}+y, n\right) - \frac{\mu^2(q)}{\phi^2(q)} g^2(y, n) \right| \leq C_1 n^2 \log^{-69} n.$$

PROOF. This follows from Lemma 4.1 in exactly the same way that Lemma 3.12 follows from Lemma 3.11.

We now assume $n \geq N_0$ throughout the rest of the proof.

By a change of variable $y=\left(x-\frac{h}{q}\right)$ we have

$$T(h, q) = \varepsilon\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y, n\right)\varepsilon(-ny)dy. \tag{A}$$

However, by Lemma 4.2

$$\left| \varepsilon\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y, n\right)\varepsilon(-ny)dy - \frac{\mu^2(q)}{\phi^2(q)} \varepsilon\left(-\frac{nh}{q}\right) \int_{-x_0}^{x_0} g^2(y, n)\varepsilon(-ny)dy \right|$$

$$\leq \int_{-x_0}^{x_0} \left| f^2\left(\frac{h}{q}+y, n\right) - \frac{\mu^2(q)}{\phi^2(q)} g^2(y, n) \right| dy \leq \int_{-x_0}^{x_0} C_1 n^2 \log^{-69} n \, dy$$

$$=2x_0C_1n^2\log^{-69}n=C_2n\log^{-54}n.$$

Now let $T_1(n)=\int_{-x_0}^{x_0}g^2(y,\ n)\varepsilon(-ny)dy$; so that by (A) and the above we have that if $(h,\ q)=1,\ q\leq\log^{15}n$ then

$$\left|T(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}T_1(n)\varepsilon\left(-\frac{nh}{q}\right)\right|\leq C_2n\log^{-54}n. \tag{B}$$

Let $T(n)=\sum_{m_1,\,m_2}\log m_1\log m_2$ (C)

with the conditions of summation $m_1\geq 2,\ m_2\geq 2$, and $m_1+m_2=n$.

By exactly the same arguments that we used in the proof of Theorem 3.3 we have

$$\frac{1}{3}n\log^{-2}n\leq T(n). \tag{D}$$

and for $(h,\ q)=1,\ q\leq\log^{15}n$

$$\left|\varepsilon\left(-\frac{nh}{q}\right)\right|\left|\frac{\mu^2(q)}{\phi^2(q)}\right|\ \left|T(n)-T_1(n)\right|\leq\frac{1}{\phi^2(q)}(2n\log^{-15}n). \tag{E}$$

Combining (E) with (B) we have for $(h,\ q)=1$ and $q\leq\log^{15}n$

$$\left|T(h,\ q)-\frac{u^2(q)}{\phi^2(q)}T(n)\varepsilon\left(-\frac{nh}{q}\right)\right|\leq C_2n\log^{-54}n+\frac{1}{\phi^2(q)}(2n\log^{-15}n); \tag{F}$$

so that adding (F) $\phi(q)$ times for some fixed $q\leq\log^{15}n$ we have:

$$\left|\sum_{\substack{0<h\leq q\\(h,q)=1}}T(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)\sum_{\substack{0<h\leq q\\(h,q)=1}}\varepsilon\left(-\frac{nh}{q}\right)\right|$$
$$\leq(C_2n\log^{-54}n)\phi(q)+\frac{1}{\phi^{4/3}(q)}(2n\log^{-15}n)\phi^{1/3}(q). \tag{G}$$

But $\phi(q)\leq\log^{15}n$ and by definition

$$\sum_{\substack{0<h\leq q\\(h,q)=1}}\varepsilon\left(-\frac{nh}{q}\right)=C_q(n)\ ;$$

so that it follows immediately from (G) that

$$\left|\sum_{\substack{0<h\leq q\\(h,q)=1}}T(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)C_q(n)\right|$$
$$\leq C_2n\log^{-39}n+\frac{1}{\phi^{4/3}(q)}(2n\log^{-10}n). \tag{H}$$

Now summing over all $q \leq \log^{15} n$ such that $(q, n)=1$ we have

$$\left| \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \sum_{\substack{0<h\leq q \\ (q,n)=1}} T(h, q) - T(n) \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) \right|$$

$$\leq (C_2 n \log^{-39} n)(\log^{15} n) + \left[\sum_{q \leq \log^{15} n} \frac{1}{\phi^{4/3}(q)} \right] (2n \log^{-10} n)$$

$$\leq C_2 n \log^{-24} n + C_3 (2n \log^{-10} n) \leq C_4 n \log^{-10} n\,;$$

since by Lemma 3.7

$$\sum_{q \leq \log^{15} n} \frac{1}{\phi^{4/3}(q)} \leq C_3 \ (C_3 \text{ independent of } n).$$

Hence this estimate combined with the hypothesis yields

$$\left| r(n) - T(n) \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) \right| \leq k(n) n \log^{-2} n + C_4 n \log^{-10} n, \qquad \text{(I)}$$

where $k(n) \to 0$. Now let

$$R(n) = \sum_{\substack{q < \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n)$$

and

$$S(n) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) D_q(n)$$

where

$$D_q(n) = \begin{cases} 1 \ \textit{if} \ (q, n)=1 \\ 0 \ \textit{if} \ (q, n)>1 \end{cases}$$

Then

$$\left| R(n) - S(n) \right| = \left| \sum_{q > \log^{15} n} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) D_q(n) \right|$$

$$\leq \sum_{\substack{q > \log^{15} n \\ q \text{ square free}}} \frac{1}{\phi^2(q)}\,;$$

since $\mu^2(q)=0$ if q is not square free, and by Lemma 3.9 if q is square free and $(q, n)=1$, then $|C_q(n)|=1$. Hence $|S(n)-R(n)| \leq C_5 \log^{-14} n$, by Lemma 3.7; so that $|S(n)-R(n)|=o(1)$.

By (D) what remains to be done is to show that $S(n)$ is uniformly bounded away from zero.

Let

$$f(q)=\frac{\mu^2(q)}{\phi^2(q)}C_q(n)D_q(n).$$

Since $\mu(q)$, $\phi(q)$, $D_q(n)$and $C_q(n)$ are all multiplicative functions of q, f is a multiplicative function of q. Also, by means of the trivial estimate on $C_q(n)$, namely n, and a direct application of Lemma 3.7 we have

$$\sum_{q=1}^{\infty}|f(q)|\leq n\sum_{q=1}^{\infty}\frac{1}{\phi^2(q)}<\infty \text{ for each } n;$$

so that by Lemma 3.8 we have for each n

$$S(n)=\prod_q\sum_{m=0}^{\infty}f(p^m).$$

But

If $m=0$, $f(p^m)=f(p^0)=f(1)=\frac{\mu^2(1)}{\phi^2(1)}C_1(n)D_1(n)=1.$

If $m=1$, $f(p^1)=f(p)=\frac{\mu^2(p)}{\phi^2(p)}C_p(n)D_p(n)=\frac{C_p(n)D_p(n)}{(p-1)^2}.$

If $m\geq 2$, $\mu(p^m)=0$; *so that* $f(p^m)=0$; so that

$$S(n)=\prod_p\left(1+\frac{C_p(n)D_p(n)}{(p-1)^2}\right).$$

Clearly, if n is even, $D_2(n)=0$; and by Lemma 3.9 we have $C_p(n)=(p-1)$ if $(p, n)>1$ and $C_p(n)=-1$ if $(p, n)=1$; so that

$$S(n)=\prod_{p>2}\left(1+\frac{C_p(n)D_p(n)}{(p-1)^2}\right)\geq\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$

$$\geq\prod_{m=2}^{\infty}\left(1-\frac{1}{m^2}\right)=\frac{1}{2}$$

so that Theorem 4.1 is now established.

If one tries to drop the condition $(q, n)=1$ in the definition of $m(n)$ above, then one is confronted with trying to show that either $S(n)-R(n)=o(1)$ or $S(n)-R(n)=o(S(n))$, but neither of these statements appears to be true if $q\leq\log^M n$ for any integer $M>0$.

Let $x_0^*=x_0n^{-\varepsilon}$ where $0<\varepsilon<1$. Let $m^*(n)$ be those points in $[x_0, x_0+1]$ which are not in any closed neighborhood of radius x_0^* about any rational number $\frac{h}{q}$ where $(h, q)=1$ and $q\leq\log^{15}n$.

Clearly,

$$m(n)\subset(m^*(n)\cup m^{**}(n))$$

where

$$m^{**}(n)=\underset{\substack{(h,q)=1\\(q,n)>1\\q\leq\log^{15}n\\0<h\leq q}}{U}\left[\frac{h}{q}-x_0^*,\ \frac{h}{q}+x_0^*\right].$$

But

$$m^*(n)\cap m^{**}(n)=\phi,$$

and

$$\int_{m^{**}(n)}\left|f^2(x,\ n)\right|dx\leq 2x_0^*n^2\log^{30}n\leq C_6(\text{n}\log^{-3}n)=o(n\log^{-2}n)\ ;$$

so that if

$$\int_{m^*(n)}\left|f^2(x,\ n)\right|dx=o(n\log^{-2}n), \tag{K}$$

then

$$\int_{m(n)}f^2(x,\ n)\varepsilon(-nx)dx=A(n)=o(n\log^{-2}n).$$

5. Some improper approaches to Goldbach's conjecture

The following deep result is due to Vinogradov and its proof can be found in [13] page 54.

THEOREM 5.1. *Suppose*

A_1 : $n\log^{-3}n<v\leq n$,

A_2 : $\log^{15}n<q\leq n\log^{-15}n$,

A_3 : $(h,\ q)=1$,

then,

A_4 : $\left|f\left(\frac{h}{q},\ v\right)\right|\leq(n\log^{-3}n)$,

Fix $\varepsilon>0$, arbitrarily small. Let $k(n)$ be a sequence of positive real numbers converging to zero. Consider

A_1^* : $k(n)n^{1/2}\log^{-1}n<v\leq n$.

A_2^* : $\log^{15}n<q\leq n^{1+\varepsilon}\log^{-15}n$.

A_3^* : $(h,\ q)=1$.

A_4^* : $\left|f\left(\frac{h}{q},\ v\right)\right|\leq k(n)n^{1/2}\log^{-1}n$.

LEMMA 5.1. *If A_1^*, A_2^* and A_3^* imply A_4^*, then (K) (of Section 4) is* *true.*

PROOF. Suppose $x\in m^*(n)$. By Lemma 3.10 there exist numbers h *and* q such that $(h, q)=1$, $q\le n^{1+\varepsilon}\log^{-15}n$ and $|qx-h|<n^{-(1+\varepsilon)}\log^{15}n$.

But then this implies that $\log^{15}n<q$; so that using A_1^* and the trivial inequality $\left|f\left(\frac{h}{q}, v\right)\right|\le v$ we have

$$\left|f\left(\frac{h}{q}, v\right)\right|\le k(n)n^{1/2}\log^{-1}n \quad (0\le v\le n).$$

Setting $y=x-\frac{h}{q}$ we have $|y|=\left|x-\frac{h}{q}\right|<\frac{n^{-(1+\varepsilon)}\log^{15}n}{q}<n^{-(1+\varepsilon)}$.

Hence by Lemma 3.1 we have

$$\left|f(x, n)\right|=\left|\varepsilon(ny)f\left(\frac{h}{q}, n\right)-2\pi iy\int_0^n \varepsilon(\mu y)f\left(\frac{h}{q}, u\right)du\right|$$

$$\le k(n)n^{1/2}\log^{-1}n+2\pi\left(\frac{n}{n^{1+\varepsilon}}\right)+k(n)n^{1/2}\log^{-1}n$$

$$\le(1+2\pi)k(n)n^{1/2}\log^{-1}n.$$

However, it is not the case that A_1^*, A_2^* and A_3^* imply A_4^*: for if one lets $v=n^{2/3}$ and $q=n$, then it is easy to see that for any $\theta\le\frac{\pi}{4}$

$$C_1\frac{n^{2/3}}{\log n}\le\cos\theta\,\Pi(n^{2/3})\le f\left(\frac{1}{n}, n^{2/3}\right) \text{ if } n\ge N_0.$$

In fact, one can construct an infinite number of counterexamples by letting $h=1$, $q=[n^{1/2+2\Delta}]$ and $v=n^{1/2+\Delta}$ where $\Delta>0$ is arbitrarily small.

The upshot of all of this is that it is not possible to establish (K) by trying to obtain the requisite estimate on $|f(x, n)|$ for $x\in m^*(n)$.

If (A_2^*) is replaced by

$$A_2^{**}: n^{\varepsilon}<q\le n\log^{-15}n$$

then we have

LEMMA 5.2. *If A_1^*, A_2^{**} and A_3^* imply A_4^*, then $A(n)=o(n\log^{-2}n)$ where $A(n)$ is defined in Section* 3.

PROOF. Same as proof of Lemma 5.1.

However, for any arbitrarily small $\Delta>0$ we can use the corresponding counterexample mentioned above to show that it is not the case that A_1^*, A_2^{**} and A_3^* imply A_4^*.

Also, in passing we note that it is easy to see that the conditions $r(n)=o(n\log^{-2}n)$ and $A(n)=o(n\log^{-2}n)$ are not compatible where $A(n)$ is that of either Section 3 or of Section 4.

Nor is it the case that

$$\int_{x_0}^{1-x_0} f^2(x, n)\varepsilon(-nx)dx=o(n\log^{-2}n),$$

for if this were true, then by (F) of Section 4 we would have

$$|r(n)-\varepsilon(-n)T(n)|\le k(n)n\log^{-2}n+C_2n\log^{-3}n,$$

where $k(n)\to 0$. And since $\varepsilon(-n)=1$ for all n, this fact together with (D) of Section 4 would imply that every sufficiently large integer can be expressed as the sum of two primes.

6. Final comments

In a forthcoming paper we plan to establish Theorem 3.3 without assuming the generalized Riemann hypothesis. To produce such a proof one must in addition to other important details carefully and delicately analyze the difficulty concerning the possible exceptional character in a satisfactory manner. It appears that all of the necessary tools to do this have been developed in [50].

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