0. Abstract

In this paper $T_0$-identification spaces are used to prove that the semi-$R_1$ separation axiom is not a generalization of the $R_1$ separation axiom and to determine conditions, which together with $R_1$, do and do not imply semi-$R_1$.

1. Introduction

Semi open sets were first introduced and investigated in 1963 [6]. Since 1963 semi open sets have been used to define and investigate many new topological properties. In 1975 semi-$T_i$, $i=0, 1, 2$, was defined by replacing the word open in the definition of $T_i$, $i=0, 1, 2$, by semi open, respectively, and it was proven that semi-$T_i$, $i=0, 1, 2$, is strictly weaker than $T_i$, $i=0, 1, 2$, respectively [7]. The semi-$R_1$ separation axiom was defined and investigated in 1978 [4]. In this paper the relationships between $R_1$ and semi-$R_1$ are investigated.

Listed below are definitions and theorems that will be utilized in this paper.

**Definition 1.1.** If $(X, T)$ is a space and $A \subseteq X$, then $A$ is semi open, denoted by $A \in \text{SO}(X, T)$, iff there exists $O \in T$ such that $O \subseteq A \subseteq O$ [6].

**Definition 1.2.** Let $(X, T)$ be a space and let $A, B \subseteq X$. Then $A$ is semi closed iff $X - A$ is semi open and the semi closure of $B$, denoted by $\text{scl } B$, is the intersection of all semi closed sets containing $B$ [1].

**Definition 1.3.** A space $(X, T)$ is $R_1$ iff for $x, y \in X$ such that $[x] \neq [y]$ there exist disjoint open sets $U$ and $V$ such that $[x] \subseteq U$ and $[y] \subseteq V$ [2].

**Definition 1.4.** A space $(X, T)$ is semi-$R_1$ iff for $x, y \in X$ such that $\text{scl } \{x\} \neq \text{scl } \{y\}$ there exist disjoint semi open sets $U$ and $V$ such that $\text{scl } \{x\} \subseteq U$ and $\text{scl } \{y\} \subseteq V$ [4].

**Definition 1.5.** Let $(X, T)$ be a space and let $R$ be the equivalence relation on $X$ defined by $xRy$ iff $[x] = [y]$. Then the $T_0$-identification space of $(X, T)$
is \((X_0, S_0)\), where \(X_0\) is the set of equivalence classes of \(R\) and \(S_0\) is the decomposition topology on \(X_0\) [8].

Note that the natural map \(P : (X, T) \rightarrow (X_0, S_0)\) is closed, opened, and 
\[ P^{-1}(P(O)) = O \text{ for all } O \in T. \]

**Definition 1.6.** A space \((X, T)\) is extremely disconnected iff for each \(O \in T, \partial O \in T\) [8].

**Theorem 1.1.** A space \((X, T)\) is \(R_1\) iff \((X_0, S_0)\) is \(T_2\) [5].

**Theorem 1.2.** If \((X, T)\) is \(R_1\), then \(X_0 = \{[x] | x \in X\}\) [3].

**Theorem 1.3.** A space \((X, T)\) is semi-\(T_2\) iff it is semi-\(R_1\) and semi-\(T_0\) [4].

**Theorem 1.4.** Every \(T_2\) space is semi-\(T_2\) [7].

**Theorem 1.5.** If \((X, T)\) is a space and \(A \subseteq X\), then \(scl A \subseteq A\) [1].

Let \(S_1\) be the statement "Every \(R_1\) space is semi-\(R_1\)."

2. Equivalent \(T_2\) condition for \(S_1\) and several applications

Let \(S_2\) be the statement "If \((X, T)\) is \(T_2\) and \(x \in X\) such that \([x] \not\in T\), then there exist disjoint open sets \(U\) and \(V\) such that \(x \in U \cap V\)."

**Theorem 2.1.** \(S_1\) iff \(S_2\).

**Proof.** Assume \(S_1\). Let \((X, T)\) be \(T_2\) and let \(x \in X\) such that \([x] \not\in T\). Let \(y \not\in X\), let \(Y = X \cup \{y\}\), and let \(S = \{O \in T | x \not\in O\} \cup \{O \cup \{y\} | x \in O \in T\}\). Then \(S\) is a topology on \(Y\) and \((Y_0, S_0)\) is homeomorphic to \((X, T)\), which implies \((Y_0, S_0)\) is \(T_2\) and \((Y, S)\) is \(R_1\). Since \([x] \not\in T\), then \(x, y = \{[y]\}_Y \not\in S\) and \(y \not\in (Y - \{y\}_Y) \cup \{x\} \subseteq SO(Y, S)\), which implies \(scl \{y\} \neq scl \{x\}\). Since \((Y, S)\) is \(R_1\), then \((Y, S)\) is semi-\(R_1\) and there exist disjoint semi open sets \(A\) and \(B\) such that \(scl \{x\} \subseteq A\) and \(scl \{y\} \subseteq B\). Let \(U, V \in S\) such that \(U \subseteq A \subseteq \bar{U}_Y\) and \(V \subseteq B \subseteq \bar{V}_Y\). Then \(x \not\in U \cup V\), which implies \(U \cup V \in T\), and since \((X, T) = (X, S_X)\), then \(x \in (\bar{U}_Y \cap X) \cap (\bar{V}_Y \cap X) = \bar{U}_X \cap \bar{V}_X\).

Conversely, suppose \(S_2\). Let \((X, T)\) be \(R_1\). Let \(x, y \in X\) such that \(scl \{x\} \neq scl \{y\}\). If \([x] \neq [y]\), then there exist disjoint open sets \(U\) and \(V\) such that \([x] \subseteq U\) and \([y] \subseteq V\), which implies \(scl \{x\} \subseteq U\) and \(scl \{y\} \subseteq V\), where \(U\) and \(V\) are disjoint semi open sets. Thus consider the case that \([x] = [y]\). Since \(scl \{x\} \neq scl \{y\}\), then \([x] \not\in T\). Since \((X, T)\) is \(R_1\), then \((X_0, S_0)\) is \(T_2\). Let \(C_x \subseteq X_0\) such that.
Then \( C_x = \{ x \} \) and since \( \mathfrak{R} \) and \( \mathfrak{S} \) in \( X_0 \) such that \( C_x \subseteq \mathfrak{R} \cap \mathfrak{S} \). Then \( P^{-1}(\mathfrak{R}) \) and \( P^{-1}(\mathfrak{S}) \) are disjoint open sets in \( X \), \( x \in P^{-1}(\mathfrak{R}) \) and \( y \in P^{-1}(\mathfrak{S}) \), which implies \( P^{-1}(\mathfrak{R}) \cup \{ x \} \) and \( P^{-1}(\mathfrak{S}) \cup \{ y \} \) are disjoint open sets. If \( x \in [\overline{y}] - \{ y \} \), then \( P^{-1}(\mathfrak{R}) \cup \{ x \} \) is semi-open and does not contain \( y \), which implies \( x \in \text{scl} \{ y \} \), and since \( \text{scl} \{ y \} \subseteq [\overline{y}] \), then \( \text{scl} \{ y \} = \{ y \} \). Similarly \( \text{scl} \{ x \} = \{ x \} \) and \( \text{scl} \{ x \} \subseteq P^{-1}(\mathfrak{R}) \cup \{ x \} \) and \( \text{scl} \{ y \} \subseteq P^{-1}(\mathfrak{S}) \cup \{ y \} \).

Theorem 2.1 can be combined with the following example to prove \( S_1 \) is false.

**Example 2.1.** Let \( N \) denote the set of natural numbers with the usual topology and let \( \beta N \) denote the Stone-Čech compactification of \( N \). Then \( \beta N \) is extremely disconnected and contains nonisolated points [8]. Thus for each nonisolated point \( x \in \beta N \), there does not exist disjoint open sets \( U \) and \( V \) such that \( x \in U \cap V \).

Example 4.1 in [7] can be used to show that semi-\( R_1 \) does not imply \( R_1 \). Thus \( R_1 \) and semi-\( R_1 \) are independent.

In [4], it was shown that the \( T_0 \)-identification space of a semi-\( R_1 \) space is semi-\( T_2 \). The fact that \( S_1 \) is false can be combined with theorem 1.1 and theorem 1.4 to show that the converse of the above statement is false.

3. \( \mathcal{I}_0 \), \( \mathcal{I}_1 \), and \( \mathcal{I}_2 \) sets

Let \( \mathcal{I}_1 = \{ P | P \text{ is a topological property such that every } R_1 \text{ space with property } P \text{ is semi-} R_1 \} \) and let \( \mathcal{I}_2 = \{ P | P \text{ is a topological property such that if } (X, T) \text{ is } T_2 \text{ and has property } P \text{, then for each } x \in X \text{ such that } \{ x \} \in T \text{ there exist disjoint open sets } U \text{ and } V \text{ such that } x \in U \cap V \} \). Since \( T_2 \subseteq \mathcal{I}_1 \) and \( T_2 \subseteq \mathcal{I}_2 \), then \( \mathcal{I}_1 \neq \mathcal{I}_2 \).

Let DNB be the property “Every point has a decreasing neighborhood base.”

**Theorem 3.1.** \( \text{DNB} \in \mathcal{I}_2 \)

**Proof.** Let \( (X, T) \) be a \( T_2 \) space with the DNB property and let \( x \in X \) such that \( \{ x \} \in T \). Net \( N \) be a decreasing nbh base of \( x \), let \( \geq \) be a well ordering of \( N \), and let \( F \) be the first element of \( N \). For each \( N \in N \), let \( \mathcal{I}_N = \{ O \in \mathcal{N} \mid 0 \leq N \} \). Then for each \( N \in \mathcal{N} \) there exists \( f_N : \mathcal{I}_N \rightarrow \mathcal{N} \times T \times T \) such that

1. \( f_N(F) = f_F(F) = (F, U_F, V_F) \), where \( U_F \) and \( V_F \) are disjoint open subsets
of $F$ and $x \in U_F \cup V_F$, and if $F < O \leq N$, then $f_N(O) = f_O(O) = (\hat{O}, U_O, V_O)$, where $\hat{O}$ is the least element of $\{W \in \mathcal{N} | W \cap O \text{ and } W \cap (\bigcup_{R < O} U_R) \cup (\bigcup_{R < O} V_R) = \emptyset \}$ and $U_O$ and $V_O$ are disjoint open subsets of $\hat{O}$ such that $x \in \bigcup_{R < O} U_R \cup \bigcup_{R < O} V_R$ and $\hat{O} = F, U_O = U_F$, and $V_O = V_F$ otherwise, and (2) $(\bigcup_{O \leq N} U_O) \cap (\bigcup_{O \leq N} V_O) = \emptyset$. The proof is by transfinite induction.

Since $(X, T)$ is $T_2$ and $\{x\} \in T$, then every nbh of $x$ contains infinitely many points, which implies there exist disjoint open sets $U_F, V_F \subseteq F$ such that $x \in U_F \cup V_F$. Then $f_F : \mathcal{F} \to \mathcal{N} \times T \times T$ defined by $f_F(F) = (F, U_F, V_F)$ satisfies the desired properties.

Assume the statement is true for all $W \in \mathcal{N}$ less than $N$. If $x \in \bigcup_{R < N} U_R \cup \bigcup_{R < N} V_R$ then $f_N : \mathcal{N} \to \mathcal{N} \times T \times T \times T$ defined by

$$f_N(O) = \begin{cases} f_O(O) \text{ if } O < N \\ f_F(F) \text{ if } O = N \end{cases}$$

satisfies the desired properties.

Thus consider the case that $x \in \bigcup_{R < N} U_R \cup \bigcup_{R < N} V_R$. Then $x \in \bigcup_{R < N} U_R$, and suppose not. Let $O \in T$ such that $x \in O$. Then there exists $W \in \mathcal{N}$ such that $W \subseteq O$. Let $P$ be the least element of $\{B \subseteq \mathcal{N} | W \cap (\bigcup_{R < B} U_R) = \emptyset \}$ and let $S$ be the immediate successor of $P$. Then $S \subseteq N$. Since $x \in \bigcup_{R < S} U_R \cup \bigcup_{R < S} V_R$, then $f_S(S) = (\hat{S}, U_S, V_S)$, where $\hat{S}$ is the least element of $\{Y \in \mathcal{N} | Y \subseteq S \text{ and } Y \cap (\bigcup_{R < S} U_R) \cup (\bigcup_{R < S} V_R) = \emptyset \}$ and $U_S, V_S$ are disjoint open subsets of $\hat{S}$ such that $x \in \bigcup_{R < S} U_R \cup \bigcup_{R < S} V_R$. Since $S \subseteq W$ or $W \subseteq S$, $\hat{S} \cap (\bigcup_{R < S} U_R) = \emptyset$, and $W \cap (\bigcup_{R < S} U_R) = \emptyset$, then $S \subseteq W$ and $V_S \subseteq S \subseteq W \subseteq O$. Hence $x \in \bigcup_{R < N} V_R$, which is a contradiction. By a similar argument $x \in \bigcup_{R < N} U_R$. Let $\hat{N}$ be the least element of $\{Y \in \mathcal{N} | Y \subseteq N \text{ and } Y \cap (\bigcup_{R < N} U_R) \cup (\bigcup_{R < N} V_R) = \emptyset \}$ and let $U_N, V_N$ be disjoint open subsets of $\hat{N}$ such that $x \in \bigcup_{R < N} U_R \cup \bigcup_{R < N} V_R$. Then $f_N : \mathcal{N} \to \mathcal{N} \times T \times T \times T$ defined by

$$f_N(O) = \begin{cases} f_O(O) \text{ if } O < N \\ (\hat{N}, U_N, V_N) \text{ if } O = N \end{cases}$$

satisfies the desired properties.

Thus by transfinite induction the statement is true for each $N \in \mathcal{N}$. Then $A = \bigcup_{N \in \mathcal{N}} U_N$ and $B = \bigcup_{N \in \mathcal{N}} V_N$ are disjoint open sets such that $x \in A \cap B$.

Let $\mathcal{S}_0 = \{ P | P \text{ is a topological property and } (X, T) \text{ has property } P \text{ iff } (X_0, S_0) \text{ has property } P \}$.
Theorem 3.2. Compactness, separability, extremely disconnectedness, and $\text{DNB}$ are elements of $\mathcal{P}_0$.

The straightforward proof is omitted.

Theorem 3.3. $\mathcal{P}_1 \cap \mathcal{P}_0 = \mathcal{P}_2 \cap \mathcal{P}_0$.

The proof is similar to that for theorem 2.1 and is omitted.

Combining theorem 3.3, theorem 3.2, theorem 3.1, and example 2.1 proves $\text{DNB} \in \mathcal{P}_1$ and compactness, separability, and extremely disconnectedness are not elements of $\mathcal{P}_1$.

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References