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A NOTE ON MILDLY COMPACT SPACES

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A space is said to be *countably compact* if every open cover admits of a finite subcover. A space X is said to be *nearly compact* [3] if every open cover admits of a finite subfamily, the interiors of closures of whose members cover X. A space X is said to be *lightly compact* [2] if every countable open cover of X admits of a finite subfamily, the closures of whose members cover the space X. These three definitions immediately suggest a fourth property, namely, every countable open cover admits of a finite subfamily, the interiors of the closures of whose members cover the space. Present paper is a study of this property, which has been designated as mildly compact. Obviously, countably compact spaces are mildly compact and mildly compact spaces happen to be lightly compact. Also every nearly compact space is a mildly compact space. It is not difficult to construct an example of a mildly compact space which is not nearly compact. The following examples will show that the other two implications are not reversible.

EXAMPLE 1. A lightly compact space is not necessarily mildly compact. Let $X=\{a, b, c_i, a_{ij}, b_{ij} : i, j=1, 2, \cdots\}$. Let each point a_{ij} and b_{ij} be isolated. Let $\{u^k(a) : k=1, 2, \cdots\}$, $\{u^k(b) : k=1, 2, \cdots\}$ and $\{u^k(c_i) : k=1, 2, \cdots\}$ be the fundamental system of neighbourhoods at a, b and c_i's respectively, where $u^k(a) = \{a, a_{ij} : i\geq k, \forall j\}$, $u^k(b)=\{b, b_{ij} : i\geq k, \forall_j\}$ and $u^k(c_i)=\{c_i, c_{ij}, b_{ij} : j\geq k\}$. Let $\mathscr{T}$ be the topology generated by this neighbourhood system. The space $(X, \mathscr{T})$ is a space with the required properties.

EXAMPLE 2. Let $Y=\{a, c_i, a_{ij} : i, j=1, 2, \cdots\}$. Then $Y\subset X$. The space $(Y, \mathscr{T}_Y)$ is a mildly compact space which is not lightly compact, where $\mathscr{T}$ is the topology for X as constructed in Example 1 and $\mathscr{T}_Y$ is the subspace topology for Y.

DEFINITIONS. The topology generated by the family $\{\text{Int Cl } G : G\in\mathscr{T}\}$ is said to be the *semi-regularization topology* of $\mathscr{T}$. This topology will be denoted by $\mathscr{T}^*$ throughout this note. A space $(X, \mathscr{T})$ is *semi-regular* if $\mathscr{T}=\mathscr{T}^*$. A subset P of a space $(X, \mathscr{T})$ is said to be *δ-open* if P is $\mathscr{T}^*$-open. A subset F of X is said to be *δ-closed* if its complement is δ-open.

THEOREM 1. *For a space $(X, \mathscr{T})$, the following are equivalent:*

(a) $(X, \mathscr{T})$ is mildly compact.

(b) Every countable δ-open cover (that is, a cover consisting of δ-open sets) admits of a finite subfamily, the interiors of the closures of whose members cover the space.

(c) Every countable regular open cover (that is, a cover consisting of regularly open sets, that is, sets G such that $G=\mathrm{Int}\ \mathrm{Cl}\ G$) admits of a finite subcover.

(d) Every countable family of regularly closed sets having the finite intersection property has non-empty intersection.

(e) Every countable family $\mathscr{F}$ of δ-closed sets having the property that for any finite subfamily $\{F_i : i=1, 2, \cdots, n\}$ of $\mathscr{F}$, $\bigcap\{\mathrm{Cl}\ \mathrm{Int}\ F_i : i=1, 2, \cdots\}\neq\phi$, has non-empty intersection.

PROOF. (a)$\Longrightarrow$(b). Obvious, since every δ-open set is open.

(b)$\Longrightarrow$(c). It can be easily proved, since every regularly open set is δ-open.

(c)$\Longrightarrow$(d). Follows immediately in view of the fact that a set is regularly open if and only if its complement is regularly closed.

(d)$\Longrightarrow$(e). The family $\{\mathrm{Cl}\ \mathrm{Int}\ F : F\in\mathscr{F}\}$ is a countable family of regularly closed sets having the finite intersection property and hence $\bigcap\{\mathrm{Cl}\ \mathrm{Int}\ F : F\in\mathscr{F}\}\neq\phi$. Since for a closed set F, $\mathrm{Cl}\ \mathrm{Int}\ F\subset F$, this implies $\bigcap\{F : F\in\mathscr{F}\}\neq\phi$. Hence the result.

(e)$\Longrightarrow$(a). Obvious.

COROLLARY 1. *A space $(X, \mathscr{T})$ is mildly compact if and only if $(X, \mathscr{T}^*)$ is mildly compact.*

DEFINITION. A space X is said to be *δ-compact* if every countable δ-open cover admits of a finite subcover.

THEOREM 2. *A space $(X, \mathscr{T})$ is δ-compact if and only if $(X, \mathscr{T}^*)$ is countably compact.*

PROOF. Obvious.

COROLLARY 2. *For a space X, the following are equivalent:*

(a) X is δ-compact.

(b) *Every countable family of δ-closed sets having the finite intersection property has non-empty intersection.*

(c) *Every sequence of δ-closed sets which has the finite intersection property has non-empty intersection.*

(d) *Every nested sequence of non-empty δ-closed sets has non-empty intersection.*

(e) *Every countable filter base has non-empty δ-adherence.*

(f) *Every sequence has a δ-cluster point.*

COROLLARY 3. *Every semi-regular space is δ-compact if and only if it is countably compact.*

COROLLARY 4. *Every δ-compact space is mildly compact.*

REMARK. It is not known whether every mildly compact space is δ-compact or not.

DEFINITIONS. A space $(X, \mathscr{T})$ is said to be an E_1*-space* [1] if every point is expressible as a countable intersection of closed neighbourhoods. A space $(X, \mathscr{T})$ is said to be *almost regular* [4] if for every point $x \in X$ and a regularly open set $G \ni x$, there exists an open set H such that $x \in H \subset \overline{H} \subset G$ or equivalently every pair of a point and a regularly closed set not containing the point can be strongly separated.

THEOREM 3. *Every mildly compact* E_1*-space is almost regular.*

PROOF. Let X be a mildly compact E_1-space. Let $y \in X$ and A be any regularly closed set not containing y. Since X is an E_1-space, there exists a countable family of closed neighbourhoods F_n of y such that $\{y\} = \bigcap \{F_n : n=1, 2, \cdots\}$. Then $\{X \sim F_n : n=1, 2, \cdots\} \cup \{X \sim A\}$ is a countable family of open sets which cover X and since X is mildly compact, there exists a finite subfamily $\{X \sim F_{n_i} : i=1, 2, \cdots, n\}$ of $\{X \sim F_n : n=1, 2, \cdots\}$ such that $X = \bigcup \{\text{Int Cl}\,(X \sim F_{n_i}) : i=1, 2, \cdots, n\} \cup \text{Int Cl}\,(X \sim A)$, since A is regularly closed, this implies $A \subset \bigcup \{\text{Int Cl}\,(X \sim F_{n_i}) : i=1, 2, \cdots, n\}$. Let $G = \bigcup \{\text{Int Cl}\,(X \sim F_{n_i}) : i=1, 2, \cdots, n\}$. Let $H = \bigcap \{\text{Int}\, F_{n_i} : i=1, 2, \cdots, n\}$. Since each F_n is a neighbourhood of y, $y \in \text{Int}\, F_n$ for all n and hence $y \in H$. Obviously, $G \cap H = \phi$ and $A \subset G$. Thus, there exist disjoint open sets G and H such that $y \in H$ and $A \subset G$. Hence X is almost regular.

COROLLARY 5. *Every mildly compact* E_1*-space is Urysohn.*

PROOF. It follows easily since every almost regular Hausdorff space is Urysohn and every E_1-space is a Hausdorff space.

DEFINITION. A subset A of a space $(X, \mathscr{T})$ is said to be *α-mildly compact* or *mildly compact* according as $(A, \mathscr{T}_A^*)$ or $(A, (\mathscr{T}_A)^*)$ is mildly compact.

Below some easily proven results about subsets are stated.

THEOREM 4. *A dense or open subset of a space is mildly compact if and only if it is α-mildly compact.*

THEOREM 5. *For regularly closed subsets, α-mild compactness implies mild compactness.*

THEOREM 6. *Every regularly closed subset of a mildly compact space is α-mildly compact and hence mildly compact.*

THEOREM 7. *If every open subset containing a dense subset A of a space X contains a mildly compact set containing A, then A is mildly compact.*

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