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A NOTE ON nl-SEMISIMPLE RING

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In a ring A, an ideal I is said to be *large* if it has nonzero intersection with every nonzero ideal, i.e. it has nonzero intersection with every nonzero principal ideal. It is known in [4] that a ring is semiprime if and only if every large ideal has zero annihilator. For a ring A, let $\Sigma=\Sigma(A)$ be the set of all non-large maximal ideals of A. We shall call a ring A nl-*semisimple* if $\Sigma(A) \neq \phi$ and $\cap\Sigma(A)=(0)$. In this paper, we will study the maximal ideal space of the underlying ring to characterize, among other things, a complete atomic Boolean algebra in terms of maximal ideals that are not large.

In what follows, A will denote a commutative semisimple ring with unity, that is, the intersection of all its maximal ideals is zero. Let $\Omega\equiv\Omega(A)$ be the space of all maximal ideals of A endowed with the Stone-topology generated by the supports $S(a)$ $(a\in A)$ where $S(a)=\{P\in\Omega \mid a\notin P\}$. It is known that the space $\Omega(A)$ is compact. Now for an element $a\in A$, we define a set $Z(a)=\Omega-S(a)$, i.e. $Z(a)=\{P\in\Omega \mid a\in P\}$. Also for an element P of Ω we define $S(P)\equiv \cup S(a)$ $(a\in P)$. We prove following lemmas.

LEMMA 1. *For each $P\in\Omega$, $P\notin S(P)$.*

PROOF. It is obvious from the definition of $S(a)$ for each $a\in A$.

LEMMA 2. *For P_1, $P_2\in\Omega$, if $P_1\neq P_2$, then $P_1\cap P_2$ is not prime.*

PROOF. Well known.

LEMMA 3. *For each $P\in\Omega$, $\cap Z(a)$ $(a\in P)$ contains at most one element.*

PROOF. Let P_1, $P_2\in\cap Z(a)$ $(a\in P)$. Suppose $P_1\neq P_2$. Since $P_i\in\cap Z(a)$ $(a\in P)$ $i=1,2$, $P\subset P_i$, i.e. $P\subset P_1\cap P_2$. Thus $P=P_1\cap P_2$. since P is maximal. By lemma 2, P is not prime. A contradiction.

THEOREM 4. *Let $P\in\Omega$. P is large if and only if $\{P\}=S(a)$ for no element $a \in A$.*

PROOF. Let P be large. Suppose there was a nonzero $b\in A$ such that $\{P\}=S(b)$. Then this implies that $b\notin P$ and $b\in P'$ for all $P'\in\Omega$ with $P'\neq P$. But bP

$\subset P$. Since an intersection of ideals is an ideal, $bP \subset P \cap \{P' | P' \in \Omega - \{P\}\} = \cap P$ $(P \in \Omega)$. But the semisimplicity of A implies that $bP=0$. Since P has zero annihilator, $b=0$. A contradiction. Thus there is no element a in A such that $\{P\} = S(a)$. Conversely, let $P \in \Omega$ and there is no $b \in A$ such that $\{P\} = S(b)$. Let $aP=0$ for an element $a \in A$. Suppose $a \neq 0$. Then $S(a) \neq \phi$. And $aP=0$ implies $S(a) \cap S(P) = \phi$ since $S(a) \cap S(a') = S(aa')$. Note that the complement of $S(P)$ with respect to Ω is $\cap Z(a)$ $(a \in P)$. By the lemma 1, $P \notin S(P)$, and by the lemma 2, the set $\cap Z(a)$ contains at most one element. Consequently, $S(a) = \{P\}$. A contradiction. Thus $a=0$. This completes the proof.

Of course, the alternation of above theorem is that $P \in \Omega$ is not large if and only if $\{P\} = S(a)$ for some $a \in A$. Now, we have the following.

COROLLARY. *If $\Sigma \neq \phi$, then the elements of Σ are the only isolated points in Ω.*

We recall that in the category of compact Hausdorff spaces and continuous maps, a space is projective if and only if it is extremally disconnected [3]. For a completely regular Hausdorff space X, βX denotes its Stone-Čech compactification. It is known in [2] that a compact space X is extremally disconnected if and only if $X = \beta S$ for every dense subspace S. Next, let Γ be a subset of Ω. We observe that, for a nonzero element $a \in A$, $S(a)$ contains an element P of Γ if and only if $a \notin P$, that is $a \notin \cap \Gamma$. Thus a set Γ is dense in Ω if and only if $\cap \Gamma = (0)$. Proofs of the next two propositions are straightforward.

PROPOSITION 5. *A ring is nl-semisimple if and only if its maximal ideal space contains a dense subset of isolated points.*

PROPOSITION 6. *A ring A is a subdirect product of the fields A/P, $P \in \Sigma(A)$ if and only if it is nl-semisimple.*

LEMMA 7. *If $\Omega(A)$ is Hausdorff, the following are equivalent:*

(1) *A is nl-semisimple and Ω is projective.*

(2) $\beta \Sigma = \Omega$.

PROOF. (1) implies (2). Since Ω is projective, it is extremally disconnected. Also $\cap \Sigma = (0)$ implies Σ is dense in Ω, and thus $\Omega = \beta \Sigma$. (2) implies (1). Since Σ is discrete, thus $\Omega (= \beta \Sigma)$ is extremally disconnected. Σ is dense in Ω. This implies $\cap \Sigma = (0)$.

Now, we recall in [5] that a compact space Y is said to be the ***free space*** of D if it is the Stone-Čech compactification of a discrete space D. In the next corollary, A^0 will denote the set of idempotents of A.

COROLLARY 1. *Let $\Omega(A)$ be zero-dimensional. The following are equivalent.*

(1) A^0 *is rationally complete and a is nl-semisimple.*

(2) Ω *is the free space of* Σ.

PROOF. Since $\Omega(A)\simeq\Omega(A^0)$, Ω is projective.

COROLLARY 2. *A Boolean algebra is atomic and complete if and only if its Stone-space is the free space of the set of maximal ideals that are not large.*

PROOF. Note that a Boolean algebra A is atomic if and only if $\Omega(A)$ contains a dense subset of isolated points. By the corollary to theorem 4, Σ is the only set of all isolated points in Ω. Thus Σ is dense in Ω, i.e. nl-semisimple. Also note that Boolean algebra A is complete if and only if $\Omega(A)$ is projective. Hence $\beta\Sigma=\Omega$. The converse is trivial.

EXAMPLES. It is obvious from proposition 5 that every atomic Boolean algebra is nl-semisimple. Now, let X be a discrete space. Then the free space βX is projective, i.e. extremally disconnected. Thus the regular open subsets of βX are closed. Let B be the Boolean algebra of all regular open subsets of βX. Then $\Sigma(B)\simeq X$, and $\Omega(B)\simeq\beta X$, and hence B is nl-semisimple. Another trivial example of nl-semisimple ring is the ring of all sequences of real numbers.

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