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LOWER AND UPPER FORMATION RADICAL OF NEAR-RINGS

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0. Abstract

In this paper we continue the study of formation radical (F-radical) classes initiated in [3]. Hereditary and stronger properties of F-radical classes are discussed by giving construction for lower hereditary, lower stronger and lower strongly hereditary F-radical classes containing a given class M. It is shown that the Baer F-radical B is the lower strongly hereditary F-radical class containing the class of all nilpotent ideals and it is the upper radical class with $\{(I, N) \mid N \in C, N \text{ is prime}\} \subset SB$ where SB denotes the semisimple F-radical class of B and C is an arbitrary but fixed class of homomorphically closed near-rings. The existence of a largest F-radical class contained in a given class is examined using the concept of complementary F-radical introduced by Scott [5].

1. Introduction

In unifying the study of various radical properties of a near-ring defined by Van der Walt, Biedleman, Laxton, Ramakotaiah and others by general radical theory one finds it difficult due to the fact that an elementwise characterization of an ideal of a near-ring is not known. Thus some of the fundamental properties like, $P(I) \subset P(N)$ fail to hold in general [4]. In [5] Scott has introduced the concept of a O-formation radical (F-radical) class to develop formation theory analogus to groups. The existence of the lower F-radical class containing a given class was shown in [3] by giving two constructions of it.

Hereditary and stronger ($P(I) \subset P(N)$) properties of F-radical classes are discussed in section 3 of the present paper. Constructions of lower hereditary, lower stronger and lower strongly hereditary F-radical classes containing a given class are given. It is shown that the Baer F-radical B is the lower strongly hereditary F-radical class containing the class of all nilpotent ideals and it is the upper radical class with $\{(I, N) \mid N \in C, N$ is prime, I is an ideal of

$N\} \subset SB$, where SB dentoes the semisimple F-radical class of B and C is an arbitrary but fixed homomorphically closed class of near-rings. An example of an F-radical is also given to show that in general an F-radical class need not be stronger. The existence of a largest F-radical class contained in a class is discussed is section 4 using the concept of complementary F-radical [5].

2. Preliminaries

In this paper a near-ring means a left near-ring satisfying $0 \cdot x = 0$ for all x. For various definitions and elementary properties of near-rings we refer to [5]. Let C be an arbitrary but fixed homomorphically closed class of near-rings and let W be the class of all ordered pairs (I, N) where $I \triangleleft N$ (I is an ideal of N) and $N \in C$. A pair (I, N) is said to be *non-zero* if $I \neq O$. For a subset M of W, I is called an *M-ideal* if $(I, N) \in M$. A pair (J, N) is said to be an *ideal of* (I, N) *in* W (denoted by $(J, N) \leq (I, N)$) if $J \subset I$. Throughout this paper any subclass M of W is assumed to contain the pair (O, N) for all $N \in C$. Moreover any two pairs (I, N) and (I', N') are identical if N and N' are isomorphic and I is isomorphic to I' under the restriction map. For the sake of simplicity, following set notations, mostly from [3], are introduced

(1) $S_1M = \{(J, N) \mid (J, N) \leq (I, N)$ and $(I, N) \in M\}$.

(2) $QM = \{(I\theta, N\theta) \mid (I, N) \in M,\ \theta$ is a homomorphism of $N\}$

(3) $EM = \{(I, N) \mid (I/J, N/J)$ and (J, N) are in M for some $J \triangleleft N\}$

(4) $GM = \{(I/I \cap J, N/I \cap J) \mid ((I+J)/J, N/J) \in M$ for I, J ideal of N in $C\}$

(5) $RM = \{(\langle J \rangle, N) \mid (J, I) \in M$ for $J \triangleleft I \triangleleft N, N \in C\}$, $\langle J \rangle$ denotes the ideal of N generated by J.

(6) $FM = S_1QGM$

(7) $IM = \{(I, U) \mid (I, N) \in M$ and $I \subset U \triangleleft N\}$

(8) $SM = \{(I, N) \mid (I, N)$ has no nonzero ideals in $M\}$, SM is called the *semisimple class* of M.

DEFINITION 2.1. ([5]) A subset P of W is said to be a *C-Formation radical class* (*F-radical class*) if it satisfies

(9) $P = S_1P = QP = GP$:

(10) Every $N \in C$ contains the unique maximal ideal $P(N)$ such that $(P(N), N) \in P$;

(11) $(N/P(N), N/P(N)) \in SP$.

The following construction of the lower c-formation radical class containing a given class M is given in [3].

Define $M_1=QM$, $\overline{M}_1=S_1QGM_1$, and for any ordinal β, $\overline{M}_\beta=S_1QGM_\beta$, where

(12) $M_\beta=\{(I, N) \mid$ for every homomorphism θ of N with $I\theta\neq 0$, $(I\theta, N\theta)$ has a non zero ideal in $\overline{M}_\alpha$ for some ordinal $\alpha<\beta\}$.

Then $L(M)=\bigcup_\beta \overline{M}_\beta$ is the smallest C-formation radical class containing M. The following two theorems are found in [3].

THEOREM 2.2. *A subset P of W is an F-radical class if it satisfies* (9) *and*

(13) *for any $(I, N)\in W$ if $(I, N)\notin P$, then there is a homomorphism θ of N with $I\theta\neq 0$ such that $(I\theta, N\theta)\in SP$.*

THEOREM 2.3. *For a class $M(\subset W)$ if $M=S_1M=QM=IM$ then $L(M)=IL(M)$ where $L(M)$ is the lower F-radical class containing M.*

3. Hereditary and stronger properties of F-radical classes

At a first glance over the definition of S_1M and hereditary property of a class of near-rings, it may look natural to call a class $M(\subset W)$ to be hereditary if $M=S_1M$. But in that case every F-radical class becomes hereditary and one arrives at an ambiguity, since there exist non-hereditary radical properties, giving rise to an F-radical class. However, it is easy to see that if P is an F-radical class then $P=IP$ if and only if $P(N)\cap I\subset P(I)$ for all ideals I of N. This suggests the following definition.

DEFINITION 3.1. A class $M(\subset W)$ is called *hereditary* if $M=IM$.

For a given class M, define $M_1=S_1QM$, $\overline{M}_1=IM_1$,

$$\overline{M}_\beta=\begin{cases}\bigcup_{\alpha<\beta}\overline{M}_\alpha & \text{if } \beta \text{ is a limit ordinal,}\\ IS_1Q\overline{M}_{\beta-1} & \text{if } \beta-1 \text{ exists,}\end{cases} \tag{14}$$

and $\overline{M}=\bigcup_\beta \overline{M}_\beta$. Then $\overline{M}=S_1\overline{M}=Q\overline{M}=I\overline{M}$, and hence by theorem 2.3 $L(\overline{M})$ is a hereditary F-radical class. Moroever if P is any hereditary F-radical class containing M, then $\overline{M}_\beta\subset P$ for all ordinal β and hence $L(\overline{M})\subset P$. Thus we have proved the following theorem.

THEOREM 3.2. *$L(\overline{M})$, $\overline{M}$ as defined above is the lower hereditary F-radical class containing M.*

For a radical class P of near-rings $P(I)$ need not be contained in $P(N)$ for all ideals I of N [4]. A radical class P is called *stronger* if $P(I)\subset P(N)\cap I$ for all ideals I of N. In general an F-radical class need not be stronger.

EXAMPLE 3.3. Let $N=\{au+bv+cw \mid a\in Z_2,\ b\in Z_3,\ c\in Z_5\}$, where Z_n denotes the set of integers modulo n. Define addition in the natural way and multiplication by

(15) $(au+bv+cw)\cdot(a_1u+b_1v+c_1w)=ac_1v$.

Then $(N,+,.)$ is a near-ring. Consider the universal class $C=\{O,\ N,\ I_1,\ I_2,\ I_3,\ K_1,\ K_3,\ N/I_2\}$, where

$$I_1=\{au+bv \mid a\in Z_2,\ b\in Z_3\},$$
$$I_2=\{0,\ v,\ 2v\},$$
$$I_3=\{bv+cw \mid b\in Z_3,\ c\in Z_5\},$$
$$K_1=\{O,\ u\},$$
$$K_3=\{aw \mid a\in Z_5\}.$$

Using theorem 2.2 it can be varified that the class $P=\{O,\ (K_1,K_1),\ (K_1,\ I_1),\ (K_1,\ N/I_2)\}$ is an F-radical class. Here $P(I_1)=K_1\not\subset O=P(N)\cap I$.

LEMMA 3.4. *For an F-radical class P, the following are equivalent.*

(i) $P(I)\subset P(N)\cap I$ *for all* $I\lhd N,\ N\in C$.

(ii) $SP=ISP$

(iii) $P=RP$,

PROOF. We first show the equivalence of (i) and (ii). Let $(J,\ W)\in SP$, $J\subset N'\lhd N$. If $(J,\ N')\notin SP$, then $(K,\ N')\in P$ for some $K\subset J$. Thus $K\subset P(N')\subset P(N)\cap N'$ and hence $(\langle K\rangle,\ N)\in P$. This is a contradiction to $(J,N)\in SP$. Conversely suppose $SP=ISP$ and I is an ideal of N. Since $((I+P(N))/P(N),\ N/P(N))\in SP$, the pair $((I+P(N))/P(N),\ (I+P(N))/P(N))=(I/I\cap P(N),\ I/I\cap P(N))\in SP$. But $(P(I),\ I)\in P$ and $P=QP$. So $((P(I)+P(N)\cap I/P(N)\cap I,\ I/P(N)\cap I)\in SP\cap P$. Hence $P(I)\subset P(N)\cap I$.

For (i)$\Longrightarrow$(iii), let $(J,\ I)\in P,\ I\lhd N$. Clearly $(\langle J\rangle,\ N)\in P$ and hence (iii) holds since $J\subset P(I)\subset P(N)$. If $J\lhd I\lhd N$, $(J,I)\in P$ and (iii) holds, then $(\langle J\rangle,\ N)\in P$. Taking $J=P(I)$ we get $P(I)\subset P(N)\cap I$. This completes the proof.

A class M is called *stronger* if $SM=ISM$. The following construction gives the lower stronger F-radical class containing a given class M.

Define $K_0=M$,

$$(16)\quad K_\beta=\begin{cases}\bigcup_{\alpha<\beta} K_\alpha & \text{if } \beta \text{ is not a limit ordinal}\\ L(RK_{\beta-1}) & \text{if } \beta-1 \text{ exists.}\end{cases}$$

and $K=\bigcup_{\beta} K_\beta$

THEOREM 3.5. *K is the lower stronger F-radical class containing the class M.*

PROOF. It is easy to see that K satisfies (9). To prove (12) let $(I, N)\in W$ such that $(I\theta, N\theta)$ has a nonzero ideal in K for all homomorphisms θ of N with $I\theta\neq 0$. Then there exists an ordinal β_0 such that $(I\theta, N\theta)$ has a nonzero ideal in K_{β_0}. Thus $(I, N)\in L(RK_{\beta_0})=K_{\beta_0+1}\subset K$. Now we observe that $K=RK$ and therefore K is stronger. If P is any other stronger F-radical class containing M, then $K_\beta\subset P$ for all ordinals β and hence $K\subset P$.

At this point it is natural to ask whether there exist a strongly hereditary F-radical class, that is, an F-radical class which satisfies both the stronger and the hereditary properties. The following construction gives an answer to this.

For a class $M(\subset W)$ define $\overline{K}_0=M$,

$$\overline{K}_\beta=\begin{cases}\bigcup_{\alpha<\beta}\overline{K}_\alpha & \text{if } \beta \text{ is a limit ordinal,}\\ L(IR\overline{K}_{\beta-1}) & \text{if } \beta-1 \text{ exists,}\end{cases}$$

and $\overline{K}=\bigcup_\beta \overline{K}_\beta$.

Then, using the argument of theorem 3.6, it follows that $\overline{K}$ is an F-radical class. Moreover if $(I, N)\in\overline{K}$ and $I\subset N'\lhd N$, then $(I, N')\in IR\overline{K}_\beta\subset\overline{K}_{\beta+1}\subset\overline{K}$. Hence $\overline{K}$ is a strongly hereditary F-radical class containing M. Thus using transfinite induction the following theorem follows.

THEOREM 3.6. *$\overline{K}$, as constructed above is the lower strongly hereditary F-radical class containing M.*

Maxson [2] has shown that the prime radical ($I(N)=\bigcap$prime ideals of N) is stronger. In [3] it was shown that the prime radical is hereditary and that the corresponding prime F-radical $B=\{(I, N)\mid I\subset I(N)=L(A)=L(Z)\}$

$=\{(I, N)\mid N\in C,\ I=I_\alpha$ for some ordinal $\alpha\}$,

where

$$A=\{(I, N)\mid I \text{ is nilpotent, } N\in C\}$$

$$Z=\{(I, N)\mid I^2=0,\ N\in C\} \text{ and}$$

$$I_\beta=\begin{cases}\bigcup_{\alpha<\beta} I_\alpha & \text{if } \beta \text{ is a limit ordinal,}\\ J & \text{if } \beta-1 \text{ exists, } J/I_{\beta-1} \text{ is the sum of all nilpotent ideals of } N/I_{\beta-1} \text{ contained in } I/I_{\beta-1}.\end{cases} \tag{17}$$

Thus the class B gives a nontrivial example of the lower strongly hereditary F-radical class containing the classes A and Z.

4. Upper F-radical

In case of associative rings [1, p. 141] the Baer radical is the lower radical for which all nilpotent rings are radical ($B(N)=N$) and it is the upper radical for which all prime rings are semisimple ($B(N)=0$). The truth of the first part of it for near-rings has been proved above while the truth of the second part follows from the following.

THEOREM 4.1. *Let P be an F-radical class such that $(I, N)\in SP$ for any ideal I of a prime near-ring N. Then $P\subset B$ (the Baer F-radical).*

PROOF. Let $(I, N)\in P$. If N has no prime ideals then $I(N)=N$ and hence $(I, N)\in B$. On the other hand if J is a nonzero prime ideal of N, then $(N/J, N/J)\in SP$. Observe that $I\not\subset J$ implies $O\neq((I+J)/J,\ N/J)\in SP\cap P$ leading to a contradiction. Thus $I\subset J$ and hence $I=I(N)$. So $(I, N)\in B$.

COROLLARY. *The Baer F-radical class B is the upper radical class such that $\{(I, N)/N\in C,\ N$ is prime$\}\subset SB$.*

In general radical theory for rings Leavitt has raised the following question: Given a class M of rings, does there exist a largest radical class contained in M?

To provide an answer in case of near-rings we use the concept of complementary F-radicals introduced by Scott [5].

DEFINITION 4.3. For an F-radical class P, the class $P'=\{(I, N)\mid I/J\cap P(N/J)=O,\ J\lhd N,\ J\subset I\}$ is called the *complementary radical* of P.

It is easy to see that P' is an F-radical class and $P'\cap P=0$. Moreover, $P\subset(P')'$ whereas equality holds in case of descending chain conditions on ideals. It can be shown that for any two F-radicals P and Q, $P\subset Q$ if and only if $Q'\subset P'$.

THEOREM 4.4. *Let M be a subclass of W and let $M_1=\{(I, N)\mid(I, N)\notin M\}$. If $M_1=S_1M_1=QM_1$, then there exist a largest radical class contained in M.*

PROOF. Let $P=L(M_1)$, the lower F-radical class containing M. Then $P'\subset M$. If Q is any other F-radical class contained in M then $(I, N)\in M_1$ implies that $(I, N)\notin Q$. Moreover, $I/J\cap Q(N/J)=O$ for all $J\subset N$, $J\subset I$. Therefore $P\subset Q'$ and hence $Q\subset(Q')'\subset P'$.

COROLLARY 4.5. *The complementary F-radical B' of the prime F-radical B*

is the upper F-radical class contained in $M=\{(I, N)|I$ *is not nilpotent*$\}$.

The class M defined in corollary 4, 5 is too large to contain B' and so we would like to see, if there exist a smaller class in which B' is still the upper radical class. Keeping this in mind, we give the following definition.

DEFINITION 4.6. An ideal I of a near-ring N is called ***hereditarily idempotent*** if $\langle J^2\rangle=J$ for all $J\lhd N$ with $J\subset I$ where $J^2=\{a\cdot b|a, b\in J\}$

THEOREM 4.7. $B'=\{(I, N)|N\in C$, *I is hereditarily idempotent*$\}$, *where B is the prime F-radical class.*

PROOF. Let $(I, N)\in W$ such that I is not hereditarily idempotent. Then $\langle J^2\rangle\neq J$ for some $J\lhd N$, $J\subset I$. So $(\bar{O})\neq J/\langle J^2\rangle\subset I/\langle J^2\rangle\cap B(I\langle J^2\rangle)$ and hence $(I, N)\notin B'$. Conversely, if $(I, N)\in W$ such that I is hereditarily idempotent in N, then $I/J\cap B(N/J)=K/J$ contains no nonzero nilpotent ideals of N/J. Therefore $(K/J)_1=(K/J)_\alpha=O$ for all ordinals α and hence $B(K/J)=O$. This completes the proof.

COROLLARY 4.8. *B' is the upper radical class contained in the class* $M=\{(I, N)|I$ *is hereditarily idempotent*$\}$.

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