ON FOULIS PAPER

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This paper is based on the results of D.J. Foulis paper 'Relative Inverses in Baer-\(\ast\)-Semigroups' [1]. We shall follow the notation and terminology given in this paper. For the sake of completeness we are giving the following definitions. A \(\ast\)-semigroup is a semigroup \(S\) with an involutorial antiautomorphism \(x \mapsto x^*\) such that

(i) \((xy)^* = y^*x^*\) and (ii) \(x^{**} = x\) for all \(x, y \in S\).

A projection in such an \(S\) is an element \(e \in S\) with \(e = e^2 = e^*\). The partially ordered set of all projections in \(S\) is denoted by \(P(S)\), the partial order being defined by \(e \leq f\) if and only if \(e = ef\) \((e, f \in P(S))\).

A Baer-\(\ast\)-semigroup is \(\ast\)-semigroup \(S\) with a two sided zero \(0\) with the property: for each element \(a \in S\) there exists a projection \(a' \in P(S)\) such that \(\{x \in S \mid ax = 0\} = a'\)'s. We define \(P'(S) = P(S)\) by the condition \(P'(S) = \{a' \mid a \in S\}\).

A projection \(e \in P(S)\) is said to be closed if \(e = e^*\). By (v) of theorem 1 [1] a projection \(e\) is closed if and only if \(e = a^\prime\) for some \(a \in S\). An element \(a \in S\) is said to be right \(\ast\)-regular in \(S\) if \(aS = (a^*)^{-}\) \(S\), \(a\) is left \(\ast\)-regular in \(S\) if \(Sa = Sa^*\). If \(a\) in \(S\) is both right and left \(\ast\)-regular in \(S\), then \(a\) is said to be \(\ast\)-regular in \(S\).

A slight different, but equivalent definition of \(\ast\)-regular in \(S\) as defined that if \(a\) is an element of the Baer-\(\ast\)-semigroup \(S\), then \(a\) is \(\ast\)-regular in \(S\) if there exists a unique element \(a^+\) in \(S\) such that \(a = aa^+a\), \(a^+ = a^+ aa^+\), \(aa^+ = (a^*)''\) and \(a^+ a = a''\). An element \(a\) in \(S\) is range closed if the condition \(g\) in \(P'(S)\) with \(g \leq d''\) and \((gd^*)'' = (d^*)''\) necessarily implies \(g = a''\). \(a^+\) is relative inverse of \(a\).

In an involution semigroup \(S\), let \(e = e^2 = e^* \in S\) and \(f = f^2 = f^* \in S\). If there exists an element \(x \in S\) such that \(x = exf\), \(x^*x = f\), \(xx^* = e\), then we say that \(e\) and \(f\) are \(\ast\)-equivalent and we write \(x : e \sim f\) and \(x\) is partially unitary element of \(S\).

In this note we give some interesting results which are consequences of the beautiful results given in [1].
THEOREM 1. If $a$ is $*$-regular element in $S$, then

$$ \text{[g'\wedge((a^+)^*)^\dagger]^\dagger} = (ga^*)' \wedge (a^*)' $$

PROOF. Let $h = (ga^*)'$. Then $h' = (ga^*)' \leq (a^*)'$. (By thm. 1 (xiv) [1]) and $hC(a^*)'$. Hence $ha = h \wedge (a^*)' a$ by lemma 5 [1].

$$ (ha)'' = [(h \wedge (a^*)') a]' $$

$$ (h(a^*)')' = [h \wedge (a^*)']' $$

$$ [(h(a^*)') a]' = [((h \wedge (a^*)') a)' a] $$

By (xii) of thm. 1 [1]

Therefore $[(g' \wedge ((a^+)')^\dagger)^\dagger] = (ga^*)' \wedge (a^*)'$. 

THEOREM 2. If $a$ is $*$-regular in a Baer-$*$-semigroup $S$, $g \leq a''$, $(ga^*)'' = (a^*)''$ and $a' \leq g \leq 1, 1 \neq g \in P'(S)$, then $(ga^+)'' = (a^*)''$.

PROOF. $a$ is $*$-regular in $S$ implies $a$ is range closed in $S$, by lemma. 8 [1]. Then there exists $g$ in $P'(S)$ such that $g = a''$. Now $ga'' = a''$.

Then $g = a''$ which gives by cor. of thm. 11 [1]

$$ (ga^+)' = (a^+)'' $$

Further by thm. 6 and cor. of thm. 11.

$$ (ga^+)'' = (a^+)'' $$

since $(ga^+)'' \leq (a^+)''$ by (xiv) of thm. 1 [1] and $(a^+)'' = (a^*)''$.

THEOREM 3. Let $a$ be $*$-regular in a Baer-$*$-semigroup $S$ and let $a^+$ be its relative inverse. Then

$$ (e' a^*)' \wedge (a^*)' = (ea^+)'' $$

for $1 \neq e \in P'(S)$.

PROOF. Let $f = ((e' a^*)'' \wedge (a^*)')' \cdots \cdots (i)$. Then $fa = ((e' a^*)' \wedge (a^*)')a$ and

$$(fa)' = (((e' a^*)' \wedge (a^*)')a)' .$$

Since $(e' a^*)'' \leq (a^*)''$ by (xiv) of thm. 1 [1], so $(e' a^*)' C((a^*)')a$ by lemma. 5 [1], we get $(fa)' = ((e' a^*)' a)'$ which gives $(fa)' = e \wedge a'' = ea''$ by thm. 6 [1] and 36.6 [1], since $a$ is range closed by lemma 8 [1]. Hence $(fa)' = ea'' = (ea^+)'' = ((ea^+)'' a)$ by cor. of thm. 11 [1] and (xii) of thm. 1 [1]. By (i) $f \leq (a^*)''$, $(ea^+)'' \leq (a^+)'' = (a^*)''$ by (v) of cor. of thm. 11 [1]. We have $f = (ea^+)''$ by thm. 6 (v) [1]. Therefore $(e' a^*)' \wedge (a^*)'' = (ea^+)''$.

THEOREM 4. Let $a$ be $*$-regular and $a^+$ be its inverse in a Baer-$*$-semigroup.
If \( a : g \sim e \) and \((e \land g) a'' = ea''\) then \( \left( (e \land g) a^+ \right)^\prime \land g' = a^'(ea^+)\). \( 1 \neq g \in P'(S) \), \( \forall e \in P'(S) \).

**PROOF.** As \((e \land g) a'' = ea''\) which gives by cor. of thm. 11 \([1]\)
\[(e \land g) a^+ a' = (ea^+) a'.\] Now by (xii) of thm. 1 \([1]\)
we have \(( ((e \land g) a^+ ) a')'' = ((ea^+) a')''\). By thm. 6 and cor. of thm. 11 \([1]\) gives \(( (e \land g) a^+ )'' = ( (ea^+ )'')\).
Since \(( (e \land g) a^+ )'' \leq (a^+ )'' \) and \((ea^+ )'' \leq (a^+ )''\)
\[( (e \land g) a^+ )'' \land g' = ( (ea^+ )'' \land g' )\]
\[( (e \land g) a^+ )'' \land g' = [ (g(a^+ ) e) ' (ea^+) ] \] by thm. 6 (i) \([1]\).
Since \( a \) is range closed by lemma 8.
Now \( aa^+ = (a^+ )'' \) by cor. of thm. 11 \([1]\), so \( aa^+ g = (a^+ ) g \) which implies
\[(a^+ g) a^* = g(a^* )''.\]
We have \( (g(a^+ ) a^*) = g(a^* )''\).
\[(g(a^+ ) a^*) a = g(a^* )'' a.\]
Therefore \( (g(a^+ )^* ) e = g(a^* )'' a \), since \( a^* a = e \)
by def. of \* -equivalent.
We get \( ( (e \land g) a^+ )'' \land g' = [ (g(a^* )'' a)' (ea^+) ] \]
\[= [ g a a^+ a ] ' (ea^+) \] by (ii) of cor. of thm. 11 \([1]\)
\[=(g a)' (ea^+ ).\]
Since \( g a = a \) (because \( a a^* = g a a^* = (g a)(g a)^* \) by \* -cancellation law \( g a = a \)),

hence \( ( (e \land g) a^+ )'' \land g' = a^' (ea^+) \)

**THEOREM 5.** Let \( a \) be \* -regular in a Baer -\* -semigroup \( S \), \( a^+ \) be its relative inverse, \( h \leq a'' \) and \((e \lor g) \leq a'' \), and \((e, f) M \forall e, f \in P'(S) \). Then \( h = (e^g)' \) if and only if

\[(ha^*)'' \land (a^* )'' = ((e \lor g)' a^* )'' a, h, g \in P'(S).\]

**PROOF.** If \( h = (e^g)' \). Then \( h = (e^g)' \). So \( (ha^*)'' a = ((e^g)' a^* )'' a \).
By (xiv) of thm. 1 \([1]\), \( h(a^* )'' \leq (a^* )'' \). So \( h(a^* )'' C(a^* )'' \).
By lemma 5 \( ( (ha^*) a = ((ha^*)'' \land (a^* )'' a. We have
\[((ha^*)'' \land (a^* )'' a = ((e \lor g)' a^* )'' a.\]
Conversely if \( ( (ha^*)'' \land (a^* )'' a = ((e \lor g)' a^* )'' a \). Since \( (ha^*)'' \land (a^* )'' \leq (a^+ )'' = (a^* )'' \) and \((e' \lor g') a^* )'' \leq (a^* )'' \) by (xiv) of thm. 1 \([1]\), hence \( (ha^*)'' \land (a^+ )'' = ((e' \lor g') a^* )'' \) by thm. 6 as \( a \) is range closed by lemma 8.
(ха)*\(a^+\) = (ха)*\(a^+\) = ((е'/г')'\(a^+\)) by 37.7 [2] and (v) cor. of thm. 11 [1]. As (ха)*\(a^+\) \(\leq\) \(a^+\), so (ха)*\(a^+\) = (ха)*". (ха)*" = ((е'/г')'\(a^+\))" by thm. 6. \(h\vee a' = (e'\vee g')\vee a'\) because \(a\) is range closed by lemma 8. We have

\((h\vee a')\wedge a" = ((e'\vee g')\wedge a')\wedge a"\).

So \(h = (e'\wedge g') = (e'g')'\) by thm. 37.7 [2] since \((e', f)M \forall e, f \in M\).

**Theorem 6.** If \(a\) is \(*\)-regular in a Baer-\(*\)-semigroup and \(a^+\) is relative inverse, then

\([((g'/a^*)a^*)'/(a^*)") a"] = [((g'a^+)"/(a^*)") a"]/(a^*)".

**Proof.** Since \((g'a^+)"/(a^*)") = (a^*)") by (xiv) of thm. 1 and (v) of cor. of thm. 11 [1], then \((g'a^+)"/(a^*)") by lem. 5,

\((g'a^+)"/(a^*)") = (((g'a^+)"/(a^*)") a")\)

\((g'a^+)"/(a^*)") = (((g'a^+)"/(a^*)") a") by (xii) of thm. 1

\((g'a^+)"/(a^*)") = (((g'a^+)"/(a^*)") a") by (i) of cor. of thm. 11.

By (xv) of Thm 1 [1], \((g'a^+)"/(a^*)") = \((g'\vee a')\wedge a"\).

Since \(a\) is range closed by lem. 8 [1].

Let \(h = (g'\wedge a^*)a^*\)'. Then \(h' \leq (a^*)")\) hence \(h' C(a^*)"\) and \(h'a = (h'\wedge (a^*)") a\) by lem. 5 [1]

Now \((h'a") = (h'(\wedge (a^*)") a")\).

So on putting the value of \(h\).

\([((g'/(a^*)a^*)a')'/(a^*)") a"] = [((g'a^+)"/(a^*)") a"]/(a^*)"\)

**Theorem 7.** Let \(a\) is \(*\)-regular in a Baer-\(*\)-semigroup, \(a^+\) is its relative inverse in \(S\), \(g \leq a^*\), \((g'a^+)"/(a^*)") \(a^* \leq g \leq 1\) and \(e C(a^*)"\). Then

(i) \(([e/(g'a^+)"/(a^*)") a^*]) = ([a^+ e])"\)

(ii) \([e/a^*]) = [((g'\vee a')\wedge (a^*)") a^*]

**Proof.** As \((g'a^+)"/(a^*)") = (a^*)"\), \((g'a^+)"/(a^*)") \(a^* \leq g \leq 1\) and \(e C(a^*)"\). Then

Hence by lem. 5 [1] we get

\([([e/(g'a^+)"/(a^*)") a^*]) = ([e/a^*]) = ([a^+ e])"\) by (xii) of thm. 1 [1] which implies

\(([e/(g'a^+)"/(a^*)") a^*]) = ([a^+ e])" = (a^+ e)"\) by (xii) of thm. 1 [1].

Proof of (ii). Since \(a\) is range closed in \(S\) by lem. 8 [1]. Hence there exists \(g\) in \(P'(S)\) such that \(g = a^"\). Now \(0 = g' \wedge g = g' \wedge a"\) which gives \(1 = g' \wedge a'\). 1
(a*)'' = (g \lor a') \land (a*)'' \text{ and so } a = [e \land (a*)'']a = ([[(g \lor a') \land (a*)'' \land e]a) \text{ by lemma 5 [1]. Further } [e(a*)''] = (ea)'' = ([[(g \lor a') \land (a*)'' \land e]a)'' = a = aa^+a = (a*)''a.

Finally we get } [e(a*)''a]'' = [(g \lor a') \land (a*)'' \land e], \text{ since } (g \lor a') \land (a*)'' \land e \leq (a*)''.

[e(a*)'']a]'' = [e(a*)'']a]'', \text{ and } (e(a*)'')'' \leq (a*)'' \text{ by (xii) and (xiv) of thm. 1 [1].}

The author is grateful to Professor D. J. Foulis for his valuable comments and suggestions.

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