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A COMMENT ON GOLDBACH'S CONJECTURE

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0. Abstract

No matter how one chooses the major arcs in the decomposition of $[x_0, x_0+1]$ it is always true with regard to the union, $m(n)$, of the corresponding minor arcs that the integal of $f^2(x, n)\ e(-nx)$ over $m(n)$ is $O(n\log^{-1}n)$. Consequently, to establish a proof of the asymptotic formulation of Goldbach's conjecture one might be tempted to take this fact as a starting point and to then concentrate the attact on trying to obtain the requisite estimate on the integral of $f^2(x,n)\ e(-nx)$ over $M(n)$, the union of a suitably chosen family of major arcs. In this paper I show that this approach is not possible.

1. Introduction

The notation here is the same as that found in [2], and a mastery of Chapter 3 in that text is a prerequisite for reading this paper.

It is generally known among serious students of the Goldbach conjecture that it is not possible to establish the asymptotic formulation of the conjecture by further refinement of the well-known techniques due to Vinogradov for estimating exponential sums.

Further, it is an easy consequence of the weak (Chebyshev) form of the prime number theorem that

P_0: No matter how one chooses the major arcs in the decomposition of $[x_0, x_0+1]$ it is always the case that

$$\int_{m(n)} f^2(x,\ n)\ \mathrm{e}(-nx)dx=O(n\ \log^{-1}n),$$

where $m(n)$ is the union of the corresponding minor arcs.

On Goldbach's Conjecture.

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Consequently, one might be tempted to take (P_0) as a starting point and to then concentrate the attact on trying to obtain the requisite estimate on the integral

$$\int_{M(n)} f^2(x, n)\, e(-nx)\, dx,$$

where $M(n)$ is the union of a family of suitably chosen major arcs.

A very careful examination of the construction presented in [2, Chapter 3] will convince the reader (see also [5]) that to do this one would have to find functions $F(n)$, $T(n)$, $v(h,q)$ and $g(y, n)$ with at least the following properties:

P_1 : $|T(n)|/(n\log^{-1}n)\longrightarrow\infty$.

P_2 : $T(n)-\int_{-x_0}^{x_0} g^2(y, n)e(-ny)dy=O(n \log^{-1}n)$.

P_3 : *If* $1\leq q\leq F(n)$, $|y|\leq x_0$ and $(h, q)=1$, then $f(h/q+y, n)-v(h, q)g(y, n)$ $=O(n \log^{-16}n)$.

2. The basic result

Although (P_1), (P_2) and (P_3) are very close to presently known results (see [5], (107), (108) and [3], Theorem 58), this approach is dashed by the following.

THEOREM. *P_0, P_1, P_2 and P_3 imply that every sufficiently large integer is the sum of two primes.*

PROOF. Assume that $n\geq N_0$. By (P_3) and the trivial inequalities $|f(x, n)|\leq n$ and $|g(y, n)|\leq n$ and on noting that if $|z|\leq c$ and $|w|\leq c$, then $|z^3-w^3|\leq 3c^2$ $|z-w|$, we have that if $q\leq F(n)$, $|y|\leq x_0$, and $(h, q)=1$, then

$|f^2(h/q+y, n)-v^2(h, q)\, g^2(y, n)|\leq C_1 n^2 \log^{-16}n$; so that

$$|T(h, q)-v^2(h, q)\, e(-nh/q)\, T_1(n)|\leq\int_{-x_0}^{x_0}| f^2(h/q+y, n)-v^2(h, q)g^2(y, n)|\, dy$$

$$\leq\int_{-x_0}^{x_0} C_1n^2\log^{-16} n\, dy=2x_0\, C_1n^2 \log^{-16} n=C_2n \log^{-1}n,$$ where

$$T(h, q)=e(-nh/q)\int_{-x_0}^{x_0} f^2(h/q+y, n)\, e(-ny)\, dy \text{ and } T_1(n)=\int_{-x_0}^{x_0} g^2(y, n)\, e(-ny)\, dy.$$

But by (P_2) $|T(n)-T_1(n)|\leq C_3 n \log^{-1} n$: so that

$$|e(-nh/q)||v^2(h,q)||T(n)-T_1(n)|\leq C_3|v^2(h,q)|n \log^{-1} n.$$

so that

$$|T(h,q)-v^2(h,q)e(-nh/q)T(n)|\leq C_2 n \log^{-1} n+C_3|v^2(h,q)|n \log^{-1} n$$

But considering the decomposition of $[x_0, x_0+1]$ into $[x_0, 1-x_0]\cup[1-x_0, 1+x_0]$ we have first by the above

$$|T(1,1)-v^2(1,1)\, e(-n)\, T(n)|\leq C_4 n \log^{-1} n$$

and then by P_0

$$\left|\int_{x_0}^{1-x_0} f^2(x,n)\, e(-nx)\, dx\right|\leq C_5 n \log^{-1} n \text{ ; so that}$$

$$|r(n)-e(-n)\, v^2(1,1)\, T(n)|\leq C_6 n \log^{-1} n.$$

But letting $y=0$ and $h=q=1$ in P_3 we see that $v(1,1)\neq 0$. Also, $e(-n)=1$ for all n : so that by P_1 we have that every sufficiently large integer can be expressed as the sum of two primes.

Of course, one could argue that it might be possible to establish (P_3) only for q such that $2\leq q\leq F(n)$, but it is very unlikely that the function f would behave differently on just one major arc.

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