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A SUFFICIENT CONDITION FOR THE INTEGRABILITY OF REES-STANOJEVIĆ SUM

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0. Abstract

We show that the condition S of Sidon is sufficient for the integrability of the limit of Rees-Stanojević cosine sums $g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\triangle a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle a_j \cos kx$.

1. Introduction

Sidon [5] introduced the following class of cosine trigonometric series: Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k \cos kx \tag{1.1}$$

be a cosine series satisfying $a_k=o(1)$, $k\to\infty$. If there exists a sequence $\langle A_k\rangle$ such that

$$A_k\downarrow 0,\ k\longrightarrow\infty, \tag{1.2}$$

$$\sum_{k=0}^{\infty}A_k<\infty, \tag{1.3}$$

$$|\triangle a_k|\leq A_k,\ k, \tag{1.4}$$

we say that (1.1) belongs to the class S or equivalently the coefficients $\langle a_k\rangle$ satisfy the condition S.

A quasi-convex null sequence satisfies the condition S if we choose

$$A_n=\sum_{m=n}^{\infty}|\triangle^2 a_m|.$$

Recently, Rees and Stanojević [4] (see also Garrett and Stanojević [3]) introduced a new class of modified cosine sums

$$g_n(x)=\frac{1}{2}\sum_{k=0}^{n}\triangle a_k+\sum_{k=1}^{n}\sum_{j=k}^{n}\triangle a_j \cos kx \tag{1.5}$$

and obtained necessary and sufficient condition for integrability of the limit of these sums.

The object of this paper is to show that the condition S is sufficient for the integrability of the limit of (1.5).

2. Lemma

The proofs of our results are based upon the following

LEMMA 1. (Fomin [1]). *If* $|c_k|\leq 1$, *then*

$$\int_0^\pi \Big|\sum_{k=0}^n c_k \frac{\sin(k+1/2)x}{2\sin x/2}\Big| dx \leq C(n+1),$$

where C *is a positive absolute constant.*

3. Results

We establish the following theorems.

THEOREM 1. *Let the sequence* $\langle a_k\rangle$ *satisfy the condition S. Then*

$$g(x)=\lim_{n\to\infty}\sum_{k=1}^n\Big[\frac{1}{2}\triangle a_k+\sum_{j=k}^n\triangle a_j \cos kx\Big]$$

exists for $x\in(0,\pi]$ *and* $\int_0^\pi |g(x)|dx\leq C\sum_{k=0}^\infty A_k$.

PROOF. We have

$$g_n(x)=\sum_{k=1}^u\Big[\frac{\triangle a_k}{2}+\sum_{j=k}^n\triangle a_j \cos kx\Big]$$

$$=\sum_{k=1}^n\frac{1}{2}\triangle a_k+\sum_{k=1}^n\sum_{j=k}^n\triangle a_j\cos kx$$

$$=\sum_{k=1}^n\frac{\triangle a_k}{2}+\sum_{k=1}^n a_k\cos kx-a_{n+1}D_n(x)+\frac{a_{n+1}}{2}$$

Making use of Abel's transformation, we obtain

$$g_n(x)=\sum_{k=1}^n\frac{\triangle a_k}{2}+\sum_{k=1}^{n-1}\triangle a_k\Big(D_k(x)+\frac{1}{2}\Big)+a_n\Big(D_n(x)+\frac{1}{2}\Big)$$

$$-a_1-a_{n+1}D_n(x)+\frac{a_{n+1}}{2}$$

$$=\sum_{k=1}^{n-1}\triangle a_k D_k(x)+a_n D_n(x)-a_{n+1}D_n(x).$$

The last two terms tend to zero as $n\to\infty$ for $x\neq 0$ and since

$$|D_k(x)|=O(1/x) \text{ if } x\neq 0 \text{ and} \sum_{k=0}^\infty|\triangle a_k|<\infty,$$

the series $\sum_{k=1}^\infty \triangle a_k D_k(x)$ converges. Hence $\lim_{n\to\infty} g_n(x)$ exists for $x=0$.

Now applications of Abel's transformation and lemma 1 yield

$$\begin{aligned}\int_0^\pi |g(x)|\,dx &= \int_0^\pi \Big|\sum_{k=1}^\infty \triangle a_k D_k(x)\Big|\,dx \\ &= \int_0^\pi \Big|\sum_{k=1}^\infty A_k \frac{\triangle a_k}{A_k} D_k(x)\Big|\,dx \\ &= \int_0^\pi \Big|\sum_{k=1}^\infty \triangle A_k \sum_{j=0}^{k} \frac{\triangle a_j}{A_j} D_j(x)\Big|\,dx \\ &\le \sum_{k=1}^\infty \triangle A_k \int_0^\pi \Big|\sum_{j=0}^{k} \frac{\triangle a_j}{A_j} D_j(x)\Big|\,dx \\ &\le C\sum_{k=1}^\infty (k+1)\triangle A_k \\ &= C\sum_{k=1}^\infty A_k.\end{aligned}$$

This completes the proof of theorem 1.

THEOREM 2. *Let $\langle a_k\rangle$ be a sequence satisfying the condition S. Then*

$$\frac{1}{x}\sum_{k=1}^\infty \triangle a_k \sin\,(k+1/2)x = \frac{h(x)}{x}$$

Converges for $x\in(0,\pi]$ *and* $\frac{h(x)}{x}\in L\;[0,\pi]$.

PROOF. Let

$$t_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle a_j \cos kx.$$

Then

$$t_n(x) = \sum_{j=1}^{n}\sum_{k=1}^{j} \triangle a_j \cos kx = \sum_{j=1}^{n} \triangle a_j\Big(D_j(x) - \frac{1}{2}\Big)$$

Thus

$$t_n(x) + \sum_{j=1}^{n} \frac{\triangle a_j}{2} = \sum_{j=1}^{n} \triangle a_j D_j(x).$$

If $g(x)$ is the same as in theorem 1, we have

$$\begin{aligned}g(x) &= \lim_{n\to\infty}\Big[t_n(x) + \sum_{j=1}^{n}\frac{\triangle a_j}{2}\Big] = \sum_{j=1}^{\infty}\triangle a_j D_j(x) \\ &= \sum_{j=1}^{\infty}\triangle a_j\,\frac{\sin(j+1/2)x}{2\sin x/2} = \frac{h(x)}{2\sin x/2}.\end{aligned}$$

According to theorem 1, $g(x)$ exists for $x\neq 0$ and

$$g(x)=\frac{h(x)}{2\sin x/2}\in L[0,\pi]$$

If $\langle a_k\rangle$ satisfies the condition S. This establishes theorem 2.

4. Remarks

Concerning the convergence of g_n in the metric space L, the author [2] has recently proved

(i) If (1.1) belongs to the class S, then $\|f-g_n\|_{L^1}=o(1)$, $n\to\infty$.

and deduced as a corollary the following theorem of Teljakovskiǐ [6] :

(ii) If (1.1) belongs to the class S, then a necessary and sufficient condition for L^1 convergence of (1.1) is $a_n \log n=o(1)$, $n\to\infty$.

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