A SUFFICIENT CONDITION FOR THE INTEGRABILITY OF REES-STANOJEVIĆ SUM

By Babu Ram

0. Abstract

We show that the condition S of Sidon is sufficient for the integrability of the limit of Rees-Stanojević cosine sums $g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle a_j \cos kx$.

1. Introduction

Sidon [5] introduced the following class of cosine trigonometric series: Let

$$a_0 + \sum_{k=1}^{\infty} a_k \cos kx$$

be a cosine series satisfying $a_k = o(1), k \to \infty$. If there exists a sequence $(A_k)$ such that

$$A_k \downarrow 0, \quad k \to \infty,$$

$$\sum_{k=0}^{\infty} A_k < \infty,$$

$$|\triangle a_k| \leq A_k, \quad k,$$

we say that (1.1) belongs to the class $S$ or equivalently the coefficients $(a_k)$ satisfy the condition $S$.

A quasi-convex null sequence satisfies the condition $S$ if we choose

$$A_n = \sum_{m=n}^{\infty} |\triangle^2 a_m|.$$

Recently, Rees and Stanojević [4] (see also Garrett and Stanojević [3]) introduced a new class of modified cosine sums

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle a_j \cos kx$$

and obtained necessary and sufficient condition for integrability of the limit of these sums.

The object of this paper is to show that the condition $S$ is sufficient for the integrability of the limit of (1.5).
2. Lemma

The proofs of our results are based upon the following

**Lemma 1.** (Fomin [1]). *If* $|c_k| \leq 1$, *then*

$$
\int_0^\pi \left| \sum_{k=0}^{n} c_k \frac{\sin(k+1/2)x}{2 \sin x/2} \right| dx \leq C(n+1),
$$

*where C is a positive absolute constant.*

3. Results

We establish the following theorems.

**Theorem 1.** *Let the sequence* $\langle a_k \rangle$ *satisfy the condition S. Then*

$$
g(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \frac{1}{2} \triangle a_k + \sum_{j=k}^{n} \triangle a_j \cos kx \right]
$$

*exists for* $x \in (0, \pi)$ *and* $\int_0^\pi |g(x)| dx \leq C \sum_{k=0}^{\infty} A_k$.

**Proof.** We have

$$
g_n(x) = \sum_{k=1}^{n} \left[ \frac{1}{2} \triangle a_k + \sum_{j=k}^{n} \triangle a_j \cos kx \right]
= \sum_{k=1}^{n} \frac{1}{2} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle a_j \cos kx
= \sum_{k=1}^{n} \frac{1}{2} \triangle a_k + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} \frac{D_n(x) + a_{n+1}}{2}
$$

Making use of Abel's transformation, we obtain

$$
g_n(x) = \sum_{k=1}^{n} \frac{1}{2} \triangle a_k + \sum_{k=1}^{n-1} \triangle a_k \left( D_k(x) + \frac{1}{2} \right) + a_n \left( D_n(x) + \frac{1}{2} \right)
- a_1 - a_{n+1} \frac{D_n(x) + a_{n+1}}{2}
= \sum_{k=1}^{n-1} \triangle a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x).
$$

The last two terms tend to zero as $n \to \infty$ for $x \neq 0$ and since

$$
|D_k(x)| = O(1/x) \text{ if } x \neq 0 \text{ and } \sum_{k=0}^{\infty} |\triangle a_k| < \infty,
$$

the series $\sum_{k=1}^{\infty} \triangle a_k D_k(x)$ converges. Hence $\lim_{n \to \infty} g_n(x)$ exists for $x = 0$. 
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Now applications of Abel’s transformation and lemma 1 yield

\[ \int_0^\pi |g(x)|\,dx = \int_0^\pi \left| \sum_{k=1}^{\infty} \triangle a_k D_k(x) \right|\,dx = \int_0^\pi \left| \sum_{k=1}^{\infty} A_k \frac{\triangle a_k}{A_k} D_k(x) \right|\,dx = \int_0^\pi \left| \sum_{k=1}^{\infty} \triangle A_k \sum_{j=0}^{\infty} \frac{\triangle a_j}{A_j} D_j(x) \right|\,dx \leq \sum_{k=1}^{\infty} \triangle A_k \int_0^\pi \left| \sum_{j=0}^{\infty} \frac{\triangle a_j}{A_j} D_j(x) \right|\,dx \leq C \sum_{k=1}^{\infty} (k+1) \triangle A_k = C \sum_{k=1}^{\infty} A_k^r. \]

This completes the proof of theorem 1.

**THEOREM 2.** Let \( \langle a_k \rangle \) be a sequence satisfying the condition S. Then

\[ \frac{1}{x} \sum_{k=1}^{\infty} \triangle a_k \sin (k+1/2)x = -\frac{h(x)}{x} \]

Converges for \( x \in (0, \pi] \) and \( \frac{h(x)}{x} \in L[0, \pi] \).

**PROOF.** Let

\[ t_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle a_j \cos kx. \]

Then

\[ t_n(x) = \sum_{j=1}^{n} \sum_{k=1}^{j} \triangle a_j \cos kx = \sum_{j=1}^{n} \triangle a_j \left( D_j(x) - \frac{1}{2} \right) \]

Thus

\[ t_n(x) + \sum_{j=1}^{n} \frac{\triangle a_j}{2} = \sum_{j=1}^{n} \triangle a_j D_j(x). \]

If \( g(x) \) is the same as in theorem 1, we have

\[ g(x) = \lim_{n \to \infty} \left[ t_n(x) + \sum_{j=1}^{n} \frac{\triangle a_j}{2} \right] = \sum_{j=1}^{\infty} \triangle a_j D_j(x) \]

\[ = \sum_{j=1}^{\infty} \triangle a_j \frac{\sin((j+1)/2)x}{2 \sin x/2} = \frac{h(x)}{2 \sin x/2}. \]

According to theorem 1, \( g(x) \) exists for \( x \neq 0 \) and
\[ g(x) = \frac{h(x)}{2 \sin x/2} \in L[0, \pi] \]

If \( \langle a_k \rangle \) satisfies the condition \( S \). This establishes theorem 2.

4. Remarks

Concerning the convergence of \( g_n \) in the metric space \( L \), the author [2] has recently proved

(i) If (1.1) belongs to the class \( S \), then \( \|f - g_n\|_1 = o(1), \ n \to \infty \).

and deduced as a corollary the following theorem of Teljakovski \[6\] :

(ii) If (1.1) belongs to the class \( S \), then a necessary and sufficient condition for \( L^1 \) convergence of (1.1) is \( a_n \log n = o(1), \ n \to \infty \).

Maharshi Dayanand University,
Rohtak-124001.
India

REFERENCES