論 文
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分布媒介定數를 갖는 原子爐의 最適制御 2部：特異攝動論에 의한 近似解

Optimal Control of a Nuclear Reactor with Distributed Parameters-Part II: Approximate Solution by Using Singular Perturbation Theory

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Abstract

A singular perturbation theory is applied to obtain an approximate solution for suboptimal control of nuclear reactors with spatially distributed parameters. The inverse of the neutron velocity is regarded as a small perturbing parameter, and the model, adopted for simplicity, is a cylindrically symmet sical reactor whose dynamics are described by the one group diffusion equation with one delayed neutron group. The Helmholtz mode expansion is used for the application of the optimal theory for lumped parameter systems to the spatially distributed parameter systems. An asymptotic expansion of the feedback gain matrix is obtained with construction of the boundary layer correction up to the first order.

1. Introduction

The one-group time dependent diffusion equation describing the kinetics of the nuclear reactor has a time-derivative term multiplied by a small coefficient that is the inverse of the neutron velocity. This small parasitic coefficient (or singular perturing parameter) is responsible for creating a "stiff" system of differential equation in the boundary layers at one or both ends of time interval considered[1]. The exact closed-form solution is developed for this singular perturbing system.[2,3] However, the machine calculation by this method meets frequently the numbers associated with the singular perturbing parameter whose magnitudes exceed the limits acceptable to the computer. In order to avoid this difficulty, the singular perturbation approach has been developed to give an asy mptotic series solution of each modal coefficient in the Helmholtz modal expansion by Asatani and his colleagues.[4] However, they limit their treatment to control of reactor whose initial state is at the steady state. Since the initial states in the general optimization problems are the dynamic states, their result are of very limited application.

In this paper, we develop the singular perturbation approach for control of nuclear reactor whose initial state is any dynamic state. The paper should be read in conjunction of Ref. 3. The authors will follow the notations of Ref. 3, and equations given there will not be repeated here unless it is necessary.

2. Derivation of Optimal Feedback Control for Each Helmholtz Mode

The state equation of the i'th Helmholtz mode and the corresponding i'th performance index in the Helmholtz modal expansion for the control problem in Ref. 3 are:

$$\frac{d\vec{\phi}(t)}{dt}=\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{a}{\epsilon} & \frac{\lambda}{\epsilon}\\ b & -\lambda\end{pmatrix}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}u(t)+\begin{pmatrix}0\\w\end{pmatrix}, \quad (1)$$

$$J=\frac{1}{2}\tilde{\vec{\phi}}(T)\overleftrightarrow{Q}\tilde{\vec{\phi}}(T)+\frac{1}{2}r\int_0^T \tilde{u}^2(t)\ dt \quad (2)$$

where the subscript i representing the modal or-

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der is omitted.

Introducing the well-known Riccati-like decoupling in the optimal control theory of lumped parameter system[5], the following control law is derived in the form of feedback type:

$$u^*(t) = -\frac{1}{r\epsilon}\overleftrightarrow{B}'\left[\overleftrightarrow{K}(t)\vec{\phi}(t) + \vec{g}(t)\right] + u_0 \tag{3}$$

where $\overleftrightarrow{K}(t)$ is a solution of the matrix Riccati differential equation,

$$\frac{d\overleftrightarrow{K}(t)}{dt} = -\overleftrightarrow{K}(t)\overleftrightarrow{A} - \overleftrightarrow{A}'\overleftrightarrow{K}(t) + \frac{1}{r\epsilon^2}\overleftrightarrow{K}(t)\overleftrightarrow{B}\overleftrightarrow{B}\overleftrightarrow{K}(t) \tag{4}$$

with

$$\overleftrightarrow{A} = \begin{pmatrix} \frac{a}{\epsilon} & \frac{\lambda}{\epsilon} \\ b & -\lambda \end{pmatrix} \tag{5}$$

and where $\vec{g}(t)$ is a solution of the following linear differential equation associated with the Riccati equation derived above:

$$\frac{dg(t)}{dt} = -\left[\overleftrightarrow{A}' - \frac{1}{r\epsilon^2}\overleftrightarrow{K}(t)\overleftrightarrow{B}\overleftrightarrow{B}'\right]\overleftrightarrow{g}(t) - \overleftrightarrow{K}(t)\left[\overleftrightarrow{B}\frac{u_0}{\epsilon} + \overleftrightarrow{W}\right] \tag{6}$$

with

$$W = \begin{pmatrix} 0 \\ w \end{pmatrix}. \tag{7}$$

The partitioning

$$\left.\begin{array}{l} \overleftrightarrow{K} = \begin{pmatrix} k_1 & k_2 \\ k_2 & k_3 \end{pmatrix} = \begin{pmatrix} \epsilon\tilde{k}_1 & \tilde{k}k_2 \\ \epsilon\tilde{k}_2 & \tilde{k}_3 \end{pmatrix} \\ \vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \epsilon\tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix} \quad \overleftrightarrow{Q} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = \begin{pmatrix} \epsilon\tilde{q}_1 & 0 \\ 0 & \tilde{q}_3 \end{pmatrix} \end{array}\right\} \tag{8}$$

leads the following Riccati equations for each element,

$$\left.\begin{array}{l} \epsilon\frac{d\tilde{k}_1}{dt} = -2a\tilde{k}_1 + \frac{1}{r}\tilde{k}_1^2 - 2\epsilon\ b\tilde{k}_2 \\ \epsilon\frac{d\tilde{k}_2}{dt} = -\lambda\tilde{k}_1 - a\tilde{k}_2 - b\tilde{k}_3 + \frac{1}{r}\tilde{k}_1\tilde{k}_2 + \epsilon\lambda k_2 \\ \frac{d\tilde{k}_3}{dt} = -2\lambda\tilde{k}_2 + 2\lambda\tilde{k}_3 + \frac{1}{r}\tilde{k}_2^2 \end{array}\right\} \tag{9}$$

and the associated linear differential equations for $g_i(t)$,

$$\left.\begin{array}{l} \epsilon\frac{d\tilde{g}_1}{dt} = -(a - \frac{1}{r}\tilde{k}_1)\ \tilde{g}_1 - b\tilde{g}_2 - \tilde{k}_1u_0 - \epsilon\tilde{k}_2w \\ \frac{d\tilde{g}_2}{dt} = -(\lambda - \frac{1}{r}\tilde{k}_2)\ \tilde{g}_1 + \lambda\tilde{g}_2 - \tilde{k}_2u_0 - \tilde{k}_3w \end{array}\right\}. \tag{10}$$

These equations should be solved under the terminal conditions,

$$\overleftrightarrow{K}(T) = \overleftrightarrow{Q} \tag{11}$$

and

$$\overleftrightarrow{g}(T) = \overleftrightarrow{Q}\overleftrightarrow{\phi}_d \tag{12}$$

3. Construction of Asymptotic Expansions

Applying the singular perturbation method developed in the lumped parameter systems[6], we seek solutions of Eq. (9) for small $\epsilon = \frac{1}{v}$ in the form

$$\tilde{k}_j = \tilde{k}_j(t,\epsilon) = \overline{k}_j(t,\epsilon) + h_j(\tau,\epsilon)\ ;\ j=1,2,3 \tag{13}$$

where $\tau = (T-t)/\epsilon$. Here, $\bar{k}_j(t,\epsilon)$ and $h_j(\tau,\epsilon)$ represent the outer solution and the boundary layer correction, respectively. The boundary layer correction can be neglected in the outer region[6]. Both $\bar{k}_j(t,\epsilon)$ and $h_j(\tau,\epsilon)$ admit asymptotic expansions in ϵ, as ϵ tends to zero, of the form

$$\bar{k}_j(t,\epsilon) = \sum_{r=0}^{\infty} k_j^r(t)\epsilon^r \tag{14}$$

$$h_j(\tau,\epsilon) = \sum_{r=0}^{\infty} h_j^r(\tau)\epsilon^r \tag{15}$$

Substituting Eqs. (13)~(15) into Eq. (9) and comparing the coefficients of like powers of ϵ up to the first order, we have the following set of equations in the outer region:

$$\left.\begin{array}{l} 0 = -2ak_1^0 + \frac{1}{r}(k_1^0)^2 \\ 0 = -\lambda k_1^0 - ak_2^0 - bk_3^0 + \frac{1}{r}k_1^0k_2^0 \\ \frac{dk_3^0}{dt} = -2\lambda k_2^0 + 2\lambda k_3^0 + \frac{1}{r}(k_2^0)^2 \end{array}\right\} \tag{16}_a$$

and

$$\left.\begin{array}{l} \left[\frac{dk_1^0}{dt}\right] = -2ak_1^1 + \frac{2}{r}k_1^0k_1^1 - \left[2bk_2^0\right] \\ \left[\frac{dk_2^0}{dt}\right] = -\lambda k_1^1 - ak_2^1 - bk_3^1 + \frac{1}{r}(k_1^1k_2^0 + k_1^0k_2^1) + \left[\lambda k_2^0\right] \\ \frac{dk_3^1}{dt} = -2\lambda k_2^1 + 2\lambda k_3^1 + \frac{2}{r}k_2^0k_2^1 \end{array}\right\} \tag{16}_b$$

where the terms in "[ ]" are the known terms in the preceding steps.

The boundary layer corrections $\sum_{r=0} h_j^r(\tau)\epsilon^r$ can be derived in the following manner. Substituting Eqs. (13)—(15) into the boundary layer equation which can be derived by transforming Eq. (9) with use made of the stretched coordinate $\tau = (T-t)/\epsilon$, we obtain the following boundary layer equations:

$$\left.\begin{aligned}
-\frac{dh_1}{d\tau}&=-2ah_1+\frac{1}{r}(h_1{}^2+2k_1h_1)-2\epsilon bh_2\\
-\frac{dh_2}{d\tau}&=-\lambda h_1-ah_2-bh_3+\frac{1}{r}(h_1h_2+k_2h_1\\
&\quad+k_1h_2)+\epsilon\lambda h_2\\
-\frac{dh_3}{d\tau}&=\epsilon\Big[-2\lambda h_2+2\lambda h_3+\frac{1}{r}(h_2{}^2+2k_2h_2)\Big]
\end{aligned}\right\}. \tag{17}$$

The resulting recursive set of equations thus becomes

$$\left.\begin{aligned}
-\frac{dh_1{}^0}{d\tau}&=-2ah_1{}^0+\frac{1}{r}h_1{}^0(h_1{}^0+2k_1{}^0)\\
-\frac{dh_2{}^0}{d\tau}&=-\lambda h_1{}^0-ah_2{}^0-bh_3{}^0+\frac{1}{r}\Big[h_1{}^0h_2{}^0+\\
&\quad h_1{}^0k_2{}^0(T)+h_2{}^0k_1{}^0(T)\Big]\\
-\frac{dh_3{}^0}{d\tau}&=0
\end{aligned}\right\}, \tag{18$_0$}$$

and for the first order

$$\left.\begin{aligned}
-\frac{dh_1{}^1}{d\tau}&=-2ah_1{}^1+\frac{1}{r}\Big[2h_1{}^0h_1{}^1+2k_1{}^0(T)h_1{}^1\Big]\\
&\quad+c_1(\tau)\\
-\frac{dh_2{}^1}{d\tau}&=-\lambda h_1{}^1-ah_2{}^1-bh_3{}^1+\frac{1}{r}\Big[h_1{}^0h_2{}^1+\\
&\quad h_1{}^1h_2{}^0+h_1{}^1k_2{}^0(T)+k_2{}^1h_1{}^0(T)\Big]+c_2(\tau)\\
-\frac{dh_3{}^1}{d\tau}&=c_3(\tau)\ \ h_1{}^1k_2{}^0(T)
\end{aligned}\right\} \tag{18$_1$}$$

where

$$\left.\begin{aligned}
c_1(\tau)&=\frac{2}{r}k_1{}^1(T)h_1{}^0-2bh_2{}^0\\
c_2(\tau)&=\frac{1}{r}\Big[k_2{}^1(T)h_1{}^0+k_1{}^1(T)h_2{}^0\Big]+\lambda h_2{}^0\\
c_3(\tau)&=\epsilon\Big[-2\lambda h_2{}^0+2\lambda h_3{}^0+\frac{1}{r}(h_2{}^{02}+2k_2{}^0h_2{}^0)\Big]
\end{aligned}\right\} \tag{19}$$

are the known terms in the preceding steps.

The solutions for these recursive sets of Eqs. (16) and (18) can be obtained in the following way. The zeroth-order equation in the outer region, Eq. (16)$_0$, we adopt

$$k_3{}^0(T)=\tilde{q}_3 \tag{20}$$

Then, Eq. (16)$_0$ can be solved, and $k_1{}^0(T)$ and $k_2{}^0(T)$ are obtained. Since the complete expansion satisfies the original terminal condition, Eq. (11), we can adopt the following terminal conditions for the zeroth-order and first-order coefficients as

$$k_j{}^0(T)+h_j{}^0(0)=\tilde{q}_j;\ j=1,2,3 \tag{21}$$

$$k_j{}^1(T)+h_j{}^1(0)=0;\ j=1,2,3 \tag{22}$$

where $\tilde{q}_j$ are given in Eq. (8) $(q_2=0)$. Since $k_j{}^0(T)'$ are already determined, $h_j{}^0(0)$ are also determined thoroughly by Eq. (21) Then, $h_j{}^0(\tau)$ are obtained by **Eq. (18)$_0$**

The terminal conditions for the first-order coefficients $k_j{}^1(t)$ and $h_j{}^1(\tau)$ are determined as follows, the procedure being peculiar to the singular perturbations. Since the boundary layer corrections are assumed to be zero in the outer region, all $h_j{}^r(\tau)$ are required to satisfy

$$\lim_{\tau\to\infty} h_j{}^r(\tau)=0. \tag{23}$$

The integration of the third equation of Eq. (18)$_1$ with use of Eq. (23) leads to

$$h_3{}^1(\tau)=\int_\tau^\infty c_3(\tau)d\tau. \tag{24}$$

Then, the terminal of the first-order coefficient of the outer expansion $k_3{}^1(T)$ can bederived by means of Eqs. (22) and (24), resulting in

$$k_3{}^1(T)=-\int_0^\infty c_3(\tau)d\tau, \tag{25}$$

under which $k_j{}^1(t)$ can be solved thoroughly by Eq. (16)$_1$. Then, we can evaluate each coefficient of the first outer expansion at the terminal time $t=T$. The values of the boundary layer correctors at the terminal time $\tau=0$ directly follow by using the relation, Eq. (22),

$$h_1{}^1(0)=-k_1{}^1(T),\ \ h_2{}^1(0)=-k_2{}^1(T) \tag{26}$$

Thus, we can carry out the computation of the first boundary layer correctors given in Eq. (18)$_1$

As to the linear differential equation for g_j, a similar procedure can be applied as follows. We seek solutions of Eq. (10) in the form

$$\tilde{g}_j=\tilde{g}_j(t,\epsilon)=\sum_{r=0}\Big[g_j{}^r(t)+f_j{}^r(\tau)\Big]\epsilon^r \tag{27}$$

where the coefficients $g_j{}^r(t)$and $f_j{}^r(\tau)$ represent the slow mode valid in the outer region and the fast dominant in the boundary layer, respectively. By the simiar procedure for $k_j{}^r(t)$ and $h_j{}^r(\tau)$, we obtain

$$\left.\begin{aligned}
0&=-(a-\frac{1}{r}k_1{}^0)g_1{}^0-bg_2{}^0-k_1{}^0u_0\\
\frac{dg_2{}^0}{dt}&=-(\lambda-\frac{1}{r}k_2{}^0)g_1{}^0+\lambda g_2{}^0-k_2{}^0u_0-k_3{}^0w
\end{aligned}\right\}, \tag{28$_0$}$$

$$\left.\begin{aligned}
\Big[\frac{dg_1{}^0}{dt}\Big]&=(-a+\frac{1}{r}k_1{}^0)\ \ g_1{}^1-bg_2{}^1\\
&\quad+\Big[\frac{1}{r}k_1{}^1g_1{}^0-k_1{}^1u_0-k_2{}^0w\ \Big]\\
\frac{dg_2{}^1}{dt}&=(-\lambda+\frac{1}{r}k_2{}^0)\ \ g_1{}^1+\lambda g_2{}^1+\Big[\frac{1}{r}k_2{}^1g_1{}^0\\
&\quad-k_2{}^1u_0-k_3{}^0w\Big]
\end{aligned}\right\} \tag{28$_1$}$$

$$-\frac{df_1^0}{d\tau}=(-a+\frac{1}{r}k_1^0(T)+\frac{1}{r}h_1^0)f_1^0-bf_2^0+\left[\frac{1}{r}h_1^0 g_1^0(T)+h_1^0u_0\right]$$
$$-\frac{df_2^0}{d\tau}=0 \qquad (29)_0$$

$$-\frac{df_1^1}{d\tau}=\left(-a+\frac{1}{r}k_1^0(T)+\frac{1}{r}h_1^0\right)f_1^1+\left[d_1(\tau)\right]$$
$$-\frac{df_2^1}{d\tau}=\left[d_2(\tau)\right] \qquad (29)_1,$$

where

$$d_1(\tau)=\frac{1}{r}\left(k_1^1(T)f_1^0+h_1^1f_1^0+h_1^0 g_1^1(T)+h_1^1 g_1^0(T)\right)-bf_2^1-h_1^1u_0-h_2^0w$$
$$d_2(\tau)=-\lambda f_1^0+\frac{1}{r}\left(h_2^0g_1^0(T)+k_2^0(T)f_1^0+h_2^0f_1^0\right)+\lambda f_2^0-h_2^0u_0-h_3^0w \qquad (30),$$

and where the terms in "[ ]" are the known terms in the preceding steps. The terminal conditions are

$$g_1^0(T)+f_1^0(0)=-\tilde{q}_1 x_d, \qquad (31)$$
$$g_2^0(T)+f_2^0(0)=-\tilde{q}_2 y_d, \qquad (32)$$
$$g_1^1(T)+f_1^1(0)=0, \qquad (33)$$
$$g_2^1(T)+f_2^1(0)=0. \qquad (34)$$

For the zeroth-order equation in the outer region, we adopt

$$g_2^0(T)=-\tilde{q}_2 y_d, \qquad (35)$$

Then, Eq. $(28)_0$ can be solved, and consequently $g_1^0(T)$ are determined. Substitutions of $g_1^0(T)$ and $g_2^0(T)$ into Eqs. (31) and (32) determine $f_1^0(0)$ and $f_2^0(0)$, respectively, and using these values, *Eq.* $(29)_0$ can be solved thoroughly.

The integration of the second equation of Eq. $(29)_1$ leads to

$$f_2^1(\tau)=\int_\tau^\infty c_2(\tau)d\tau \qquad (36)$$

since $f_2^1(\infty)=0$ Then, $g_2^1(T)$ can be determined by Eq. (34) under which Eq. $(28)_1$ can be solved thoroughly. Accordingly, we can evaluate $g_1^1(T)$ and $g_2^1(T)$, which, in turn, determine $f_1^1(0)$ and $f_2^1(0)$ by Eqs. (33) and (34), respectively. Thus, we can carry out the computation of the first boundary layer correctors given in Eq. $(29)_1$.

The above procedure determines the total system of the zeroth and first orders completely. The similar result can be derived for the higher order systems by using the same algorithm. However, the calculation is too cumbersome compared to the exact closed-form solution given in Part I of this paper.[3]

4. Remarks

We have derived an approximate solution of an optimal problem, with a terminal cost, arising in distributed parameter nuclear reactors whose initial states are any dynamic states. The method is based on the singular perturbation theory. The sample results and comparisions with the exact closed-form solution in Part I will be given in Part III.

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