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—Research Paper—

An Order Level Inventory Model for Deteriorating Items with Power Pattern Demand

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Abstract

An order level inventory model is developed for deteriorating items. The demand during prescribed scheduling period is constant and deterministic in which the demand follows power pattern. Deterioration is assumed to be a constant fraction of the on hand inventory. The expression for the optimal order level is developed and an example is given to illustrate the model.

Introduction

A number of authors have developed mathematical models of inventory for perishable or deteriorating items to describe optimal policies.

Ghare and Schrader (2) have developed an EOQ model for exponentially decaying inventory; Covert and Philip (1) have developed an EOQ model for items with a variable rate of deterioration by taking two parameter weibull distribution for the time to deterioration of an item; Philip (7) has developed an EOQ model by taking a three parameter weibull distribution for the time to deterioration of an item; Misra (4) has developed an optimum production lot size model for items with constant and variable rate of deterioration; Shah (8) has developed a model in which shortages are permissible and which generalizes the work of Ghare and Schrader and that of Covert and Philip;

Tadikamalla (10) has developed an EOQ model with no shortages for items with a variable rate of deterioration by taking gamma distribution for the time to deterioration of an item and compare the result with Covert and Philip's (1); Upenda Dave (11) has developed an order level lot size model for exponentially decaying inventory in which the time variable is assumed to be a discrete one.

Nahmias and Wang (6) have developed a heuristic lot size reorder point model which allows for random demand, exponential decay and a positive lead time for ordering;

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Shah and Jaiswal (9) have developed a lot size reorder point model with a periodic review and stochastic demand and with a constant or a variable rate of deterioration; Jani and Jaiswal and Shah (3) have extended the results of Shah and Jaiswal's (9) in the case of constant rate of deterioration.

In this paper an order-level inventory model is developed for deteriorating items in which demand follows power pattern.

The solution for the model is derived and an example is also given to illustrate the model.

Development of the Model

If we consider a period of time over which a demand of size A occurs, there are numerous ways by which quantities are taken out of inventory. These different ways by which demand occurs during a period will be referred to as demand pattern. We can consider a general class of pattern which can be represented by;

$$I(t)=S-A\left(\frac{t}{T}\right)^{\frac{1}{n}} \tag{1}$$

Where $I(t)$=the amount in inventory at time $t(0\leq t\leq T)$

S=the amount inventory at the beginning of the period

n=the demand pattern index $(0<n<\infty)$

A=the demand size during period T

As in equation (1), when n is larger than 1, a larger portion of demand occurs toward the beginning of the period. When n is smaller than 1, a larger portion of demand occurs at the end of the period.

The mathematical model is developed with the following assumptions;

i) The inventory of the system is replenished regularly at the end of the scheduling period of t_p time units. At the end of the scheduling period the inventory is ordered up to the level of S. t_p is assumed to be a prescribed constant.

ii) The demand during any scheduling period is deterministic and constant amount A.

iii) Leadtime for replenishment is assumed to be zero.

iv) The unit cost C in dollars per unit, the inventory holding cost C_1 in dollars per time per unit and the shortages cost C_2 in dollars per time per unit are known and constant during the scheduling period.

v) The demand has the power pattern demand.

vi) The items in inventory are subject to constant rate of deterioration.

i.e. $g(t)=\lambda\exp(-\lambda t)\ (\lambda>0,\ t>0)$.

vii) There is no repair or replenishment of deteriorated items during scheduling period under consideration.

Let Q denote the amount that equal to the demand plus deterioration during prescribed scheduling period t_p. Then optimal order level S lies in the range of $0\leq S\lessapprox Q$. Suppose that the system carries positive inventory during $(0,\ t_1)$ and shortages occur during

(t_1, t_p). When the demand is subject to power pattern demand, the depletion of inventory due to demand during small time interval dt is expressed as follows;

$$-dI(t)=\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}} t^{\frac{1}{n}-1} dt \tag{2}$$

The differential equation describing the system is given by;

$$-dI(t)=I(t)\lambda dt+\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}} t^{\frac{1}{n}-1} dt(0\leq t\leq t_1)$$

$$-dI(t)=\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}} t^{\frac{1}{n}-1} dt(t_1\leq t\leq t_p) \tag{3}$$

Using the boundary conditions; $I(0)=S$, $I(t_1)=0$; the solution of the above differential equations is given by;

$$I(t)=\exp(-\lambda t)\left[\int_0^t -\frac{A}{n}\left(\frac{1}{t}\right)^{\frac{1}{n}} y^{\frac{1}{n}-1} \exp(\lambda y)dy+S\right](0\leq t\leq t_1) \tag{4}$$

$$I(t)=A\left[\left(\frac{t_1}{t}\right)^{\frac{1}{n}}-\left(\frac{t}{t_p}\right)^{\frac{1}{n}}\right](t_1\leq t\leq t_p)$$

where

$$S=\int_0^{t_1}\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}} y^{\frac{1}{n}-1}\exp(\lambda y)dy \tag{5}$$

The total amount of storage during a scheduling perion, I_1, is given by;

$$I_1=\int_0^{t_1} I(t)dt \tag{6}$$

The total amount of shortages during a scheduling period, I_2, is given by;

$$I_2=\int_{t_1}^{t_p}[-I(t))dt]$$

$$=A\left(\frac{1}{t_p}\right)^{\frac{1}{n}}\left(-t_p t_1^{\frac{1}{n}}+\frac{n}{n+1}t_p^{\frac{1}{n}+1}+\frac{1}{n+1}t_1^{\frac{1}{n}+1}\right) \tag{7}$$

The amount of deterioration during a scheduling period, D, is given by;

$$D=S-[\text{demand during } (0,\ t_1)]=S-A\left(\frac{t_1}{t_p}\right)^{\frac{1}{n}} \tag{8}$$

From the equations (6), (7) and (8) the total cost per unit time is as follows;

$$C(t_1)=\frac{C_1}{t_p}I_1+\frac{C_2}{t_p}I_2+\frac{C}{t_p}D$$

$$=\frac{C_1}{t_p}\int_0^{t_1} I(t)dt+\frac{C}{t_p}\left[S-A\left(\frac{t_1}{t_p}\right)^{\frac{1}{n}}\right]$$

$$+C_2A\left(\frac{1}{t_p}\right)^{\frac{1}{n}+1}\left(-t_p t_1^{\frac{1}{n}}+\frac{n}{n+1}t_p^{\frac{1}{n}+1}+\frac{1}{n+1}t_1^{\frac{1}{n}+1}\right) \tag{9}$$

Let

$$F(t)=\int_0^t y^{\frac{1}{n}-1}e^{\lambda y}dy$$

$$G(t)=\frac{F(t)}{n}$$

and

$$H=-t_p t_1^{\frac{1}{n}}+\frac{n}{n+1}t_p^{\frac{1}{n}+1}+\frac{1}{n+1}t_1^{\frac{1}{n}+1}$$

Then

$$\int_0^{t_1} e^{-\lambda t}F(t)dt=\frac{1}{\lambda}\left[nt_1^{\frac{1}{n}}-e^{\lambda t_1}F(t_1)\right]$$

and

$$G(t)=t^{\frac{1}{n}}e^t-\lambda\int_0^t y^{\frac{1}{n}}e^{\lambda y}dy$$

Equation (9) can be simplified and is given by;

$$C(t_1)=\left(\frac{1}{t_p}\right)^{\frac{1}{n}+1}\left\{[G(t_1)-t_1^{\frac{1}{n}}]\left(\frac{C_1A}{\lambda}+CA\right)+C_2AH\right\} \quad (10)$$

When $\lambda=0$ there is no deterioration and equation (9) reduce to those of the corresponding model for nondeteriorating items

$$C(S)=(C_1+C_2)\frac{S(S/A)^n}{n+1}-C_2S+\frac{C_2An}{n+1}$$

which is the same as that given by Naddor (5).

To find the optimal value of t_1 we differentiate $C(t_1)$ with respect to t_1 and set equal to zero solving for t_1.

Thus

$$\frac{dc(t_1)}{dt_1}=\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}+1}t_1^{\frac{1}{n}-1}D(t_1) \quad 0<t_1<t_p \quad (11)$$

where

$$D(t_1)=\left\{\left(C+\frac{C_1}{\lambda}\right)[\exp(\lambda t_1)-1]+C_2(t_1-t_p)\right\} \quad (12)$$

Since $D(t_1)$ is the continuous and monotonically increasing function and $D(0)<0$ and $D(t_p)>0$ for positive values of C, C_1, and C_2, there exists an unique value of t_1, t_1^* which satisfies $D(t_1^*)=0$.

Note that

$$\frac{d^2C(t_1)}{dt^2_1}=\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}+1}\left(\frac{1-n}{n}\right)t_1^{\frac{1}{n}-2}D(t_1)+\frac{A}{n}\left(\frac{1}{t_p}\right)^{\frac{1}{n}+1}\left[\lambda\left(\frac{C_1}{\lambda}+C\right)e^{\lambda t_1}+C_2\right]$$

and for t_1^*

$$\left.\frac{d^2C(t_1)}{dt^2_1}\right|_{t_1=t^*}>0.$$

From the above observation, t_1^* is the global minimum point and once t_1^* is found, S^* can be obtained from equation (5).

The direct solution of equation (12) is impossible but can be obtained numerically.

If we express the exponential term in equation (12) as an infinite series, we obtain following equation;

$$\left(C+\frac{C_1}{\lambda}\right)\left(\sum_{k=0}^{\infty}\frac{(\lambda t_1)^k}{k!}-1\right)+C_2(t_1-t_p)=0 \quad (13)$$

Ignoring terms with third and higher order powers of t_1 under the assumption that $\lambda t_1 \ll 1$, equation (13) become

$$(C_1+C\lambda)\lambda t_1^2+2t_1(C_1+C_2+C\lambda)-2C_2t_p=0 \tag{14}$$

Equation (14) can be solved for the optimal value of t_1. This gives;

$$t_1^*=\frac{-(C_1+C_2+C\lambda)+\sqrt{(C_1+C_2+C\lambda)^2+2\lambda C_2t_p(C_1+C\lambda)}}{(C_1+C\lambda)\lambda} \tag{15}$$

In equation (12) as λ approaches zero, t_1 approaches

$$\frac{C_2}{C_1+C_2}t_p$$

which is the optimal value of t_1 when deterioration do not occur. Again we obtain the optimal order level using equation (5).

When demand occurs entirely at the beginning of the scheduling period corresponding to the pattern index $n=\infty$ the optimal order level should equal to the demand size regardless of the decaying parameter and there should be no costs since no inventory would be carried and neither shortage nor depletion would occur. Similarly, when all demand occurs at the end of the period corresponding to $n=0$, the order level should be zero and there should be no cost.

In order to substantiate the above arguments, we need to recognize the following identity.

$$\lim_{n\to\infty} G(t)=1 \text{ for } t>0$$

Proof is as follow.

$$\begin{aligned}\lim_{n\to\infty} G(t) &= \lim_{n\to\infty}[t^{\frac{1}{n}}e^{\lambda t}-\lambda E(0)] \text{ where } E(0)=\int_0^t y^{\frac{1}{n}}e^{\lambda y}dy\\ &= e^{\lambda t}-\lim_{n\to\infty} E(0)\\ &= e^{\lambda t}-\lim_{n\to\infty}\lim_{a\to 0} E(a)\\ &= e^{\lambda t}-\lim_{n\to\infty}\lim_{a\to 0} E(a) \text{ by the uniform convergence property of } E(a)\\ &= e^{\lambda t}-\lim_{a\to 0}(e^{\lambda t}-e^{\lambda a})=1.\end{aligned}$$

Using equation (5), (9) and (10) it can be shown that

i) $\lim_{n\to\infty} S=A$ and $\lim_{n\to\infty} C(t_1)=0$,

ii) $\lim_{n\to 0} S=0$ and $\lim_{n\to 0} C(t_1)=0$.

From $D(t_1)=0$ in equation (12) it can be verified that as λ increases the optimal value of t_1 decreases.

A numerical example

To illustrate the model a hypothetical system is assumed with the following constants;

$A=200$ units/month

$C=\$32.4$/unit

$C_1=\$1.3$/unit/month

$C_2=\$6.48$/unit/month

$t_p=1$ month

Two different t_1 values, one obtained from the approximate solution of equation (15) and the other from the computer program using equation (13) are given in (table-1) for comparative purpose and the approximation is reasonably accurate.

Under the pattern index n, there are two values in each box, the upper one showing the optimal order level S and the lower one the corresponding cost $C(t_1)$. For instance, when $\lambda=0.1$ and $n=10$, $S=190.4$ and cost= $61.74.

As have been discussed the following results can be recognized.

i) As λ increases, the total cost increases and t_1^* decreases.

ii) As n increases the optimal order level approaches to A and the total cost approaches to zero.

iii) As n decreases to zero, the optimal order level approaches to zero and so does the total cost.

(Table 1) Optimal order level and cost

λ	t_1^* from Eq. (15)	t_1 from Eq. (13)	n 0.01	0.1	1.0	10.0	100.0
0.001	0.829	0.828	0.00 12.83	30.4 99.91	165.7 111.2	196.3 21.99	199.6 2.42
0.01	0.798	0.798	0.00 12.83	21.1 105.5	160.2 130.8	195.7 26.25	199.6 2.89
0.1	0.581	0.580	0.00 12.83	0.91 117.3	119.5 270.4	190.4 61.74	199.0 6.89
0.2	0.443	0.443	0.00 12.83	0.063 117.8	92.6 358.5	185.9 90.84	198.6 10.25
0.5	0.258	0.257	0.00 12.83	0.00028 117.8	54.9 478.5	176.7 147.2	197.6 17.00

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