INFINITESIMAL VARIATIONS OF SUBMANIFOLDS
OF A KAHLERIAN MANIFOLD

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0. Introduction

Infinitesimal variations of submanifolds of Riemannian and general metric manifolds have been studied by Davies [1], Dienes [2], Hayden [3], Schouten and van Kampen [4] and one of the present authors [5], [6].

Recently, Ki, Okumura and one of the present authors [7] studied infinitesimal variations of invariant submanifolds of a Kaehlerian manifold, and the present authors [8] studied those of anti-invariant submanifolds.

The main purpose of the present paper is to study infinitesimal variations of generic and CR submanifolds of a Kaehlerian manifold.

In §1, we quote some formulas in the theory of submanifolds of a Kaehlerian manifold and in §2 we define and study invariant, anti-invariant, generic and CR submanifolds.

In §3, we obtain rather general formulas for infinitesimal variations of submanifolds of a Kaehlerian manifold and in the last §4, we study invariant, anti-invariant, generic and CR variations.

1. Submanifolds of a Kaehlerian manifold

Let $M^{2m}$ be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U_i; x^i\}$, and $F^h_i$ and $g_{ij}$ the almost complex structure tensor and the almost Hermitian tensor of $M^{2m}$ respectively, where, here and in the sequel, the indices $h, i, j, k...$ run over the range $\{1', 2', ..., (2m)\}$. Then we have

\begin{align*}
(1.1) & \quad F^h_iF^i_h = -\delta^h_j, \\
(1.2) & \quad F^h_iF^i_s g_{is} = g_{j}j
\end{align*}

and

\begin{equation}
(1.3) \quad \nabla_j F^h_i = 0,
\end{equation}

where $\nabla_j$ denotes the operator of covariant differentiation with respect to $g_{ji}$.

Let $M^n$ be an $n$-dimensional Riemannian manifold covered by a system of
coordinate neighborhoods \{V; y^a\} and $g_{ab}$ the metric tensor of $M^n$, where, here and in the sequel, the indices $a, b, c, \ldots$ run over the range \{1, 2, ..., $n$\}. We assume that $M^n$ is isometrically immersed in $M^{2m}$ by the immersion $i: M^n \rightarrow M^{2m}$ and identify $i(M^n)$ with $M^n$. We represent the immersion $i: M^n \rightarrow M^{2m}$ locally by

$$x^h = x^h(y^a)$$

and put

$$B^h_i = \partial_b x^h (\partial_b = \partial/\partial y^b).$$

These $B^h_i$ are $n$ linearly independent vectors tangent to the submanifold $M^n$. Since the immersion is isometric, we have

$$g_{cb} = g_{ij} B^i_c B^j_b.$$

We denote by $C^h_y$ $2m-n$ mutually orthogonal unit normals to $M^n$, where, here and in the sequel, the indices $x, y, z$ run over the range \{n+1, ..., 2m\}. Then the equations of Gauss are given by

$$\nabla_c B^h_i = h_{cb} C^h_z$$

and those of Weingarten by

$$\nabla_c C^h_y = - h_{cz} B^h_a,$$

where $\nabla_c$ denotes the operator of van der Waerden–Bortolotti covariant differentiation along $M^n$ and the second fundamental tensors $h_{cb}^z$ and $h_{cz}^b$ are related by

$$h_{cz}^b = h_{cb}^y g_{by} = h_{cb}^y g_{by} g_{zy},$$

g_{by} being contravariant components of $g_{ba}$ and $g_{zy}$ the covariant components of the metric tensor of the normal bundle.

Now decomposing $F^h_i B^i_j$ and $F^h_i C^i_y$ into tangential and normal parts respectively, we have equations of the form

$$F^h_i B^i_j = f^a_i B^i_a - f^a_j C^a_i$$

and

$$F^h_i C^i_y = f^a_j B^i_a + f^a_y C^a_i.$$

Since $F_{ji} = - F_{ij}$ where $F_{ji} = F^l_j g_{li}$, we have

$$f_{by} = f_{yb},$$

where $f_{by} = f^b_a g_{zy}$ and $f_{yb} = f^b_y g_{eb}$.

Applying $F$ to the both sides of (1.10) and (1.11), using (1.10) and (1.11) and comparing the tangential and normal parts, we find

$$f_x f_x^a - f_y f_y^a = - \delta^a_x,$$

$$f_x f_x^a + f_y f_y^a = 0,$$

$$f_y f_x^a + f_y f_y^a = 0,$$
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(1.16) \[-f^y f^x + f^x f^y = -\delta^y_x.\]

Differentiating (1.10) and (1.11) covariantly along \( M^n \), using (1.10) and (1.11), and comparing tangential and normal parts, we find

(1.17) \[ \nabla_c f^u = h^{uc} f^u, \]

(1.18) \[ \nabla_c f^\pi = h^{ce} f^{\pi} - h^{\pi c} f^e, \]

(1.19) \[ \nabla_c f^x = -h^{\pi c} f^x + h^{ce} f^\pi, \]

(1.20) \[ \nabla_c f^\pi = h^{\pi c} f^x - h^{ce} f^\pi. \]

2. Invariant, anti-invariant, generic and CR submanifolds

When the tangent space of \( M^n \) is invariant under the action of \( F \), the submanifold \( M^n \) is said to be invariant or complex in \( M^{2m} \). A necessary and sufficient condition for \( M^n \) to be invariant is that

(2.1) \[ f_b^s = 0 \]

in (1.10).

When the transform of the tangent space of \( M^n \) by \( F \) is always normal to \( M^n \), the submanifold \( M^n \) is said to be anti-invariant or totally real in \( M^{2m} \) [9]. A necessary and sufficient condition for \( M^n \) to be anti-invariant is that

(2.2) \[ f_b^s = 0 \]

in (1.10).

When the transform of the normal space of \( M^n \) by \( F \) is always tangent to \( M^n \), the submanifold \( M^n \) is said to be generic in \( M^{2m} \) [10]. A necessary and sufficient condition for \( M^n \) to be generic is that

(2.3) \[ f_x^s = 0 \]

in (1.11).

When there exist complementary distributions \( L \) and \( M \) in the tangent space of the submanifold \( M^n \) and \( L \) is invariant under the action of \( F \) and \( M \) is transformed into a space normal to \( M^n \), the submanifold \( M^n \) is called a CR submanifold [11].

We denote by \( l_b^a \) and \( m_b^a \) the projection operators on \( L \) and \( M \) respectively. Then we have

(2.4) \[ l^2 = l, \quad m^2 = m, \quad lm = ml = 0, \quad l + m = 1. \]

First of all, we have from (1.10)

\[ F^h_i (B^l_b l^i_c) = (f^a_l c^b) B^h_a - (f^s_l c^b) C^h_s, \]

from which, the distribution \( L \) being invariant under the action of \( F \), we have
(2.5) \[ m_\epsilon f_\epsilon \epsilon^\epsilon = 0 \]

and

(2.6) \[ f_\epsilon \epsilon^\epsilon = 0. \]

We also have from (1.10)

\[ F^h_i (B^i_\theta^j m_\epsilon^b) = (f_\theta^a m_\epsilon^b) B^h_a - (f_\theta^a m_\epsilon^b) C^h_a, \]

from which, the distribution \( M \) being anti-invariant under the action of \( F \), we have \( f_\theta^a m_\epsilon^b = 0 \) and consequently

(2.7) \[ f_\theta^a \epsilon^\epsilon = f_\epsilon^a. \]

Thus transvecting (1.14) with \( \epsilon_i^k \) and using (2.6) and (2.7), we find

(2.8) \[ f_\theta^a \epsilon^\epsilon = 0 \]

and consequently

(2.9) \[ f_\theta^a \epsilon^\epsilon = 0. \]

Conversely, suppose that (2.8) is satisfied. Then we have, from (1.13)

\[ f_\epsilon^a f_\epsilon^a + \epsilon_\epsilon^a = 0, \]

which shows that \( f_\epsilon^a \) defines an \( f \)-structure. Thus, if we put

\[ \theta_\epsilon^a = -f_\epsilon^a \epsilon^a, \quad \mu_\epsilon^a = f_\epsilon^a \epsilon^a + \delta_\epsilon^a \]

we can easily see that \( l \) and \( m \) are complementary projection operators defining distributions \( L \) and \( M \) respectively.

We can verify also that \( l \) and \( m \) thus defined satisfy

\[ m_\epsilon f_\epsilon \epsilon = 0, \quad f_\epsilon \epsilon^\epsilon = 0 \]

because of (2.8). Thus we have from (1.10)

\[ F^h_i (B^i_\theta^j m_\epsilon^b) = (f_\theta^a m_\epsilon^b) B^h_a, \]

which shows that \( L \) is invariant under the action of \( F \) because of \( m_\epsilon f_\epsilon \epsilon^\epsilon = 0 \). We also have from (1.10)

\[ F^h_i (B^i_\theta^j m_\epsilon^b) = - (f_\theta^a m_\epsilon^b) C^h_a, \]

because of \( f_\theta^a m_\epsilon^b = 0 \), which shows that \( M \) is anti-invariant under the action of \( F \). Thus we have

**Proposition 2.1.** A necessary and sufficient condition for a submanifold \( M^n \) in \( M^{2m} \) to be a CR submanifold is that \( f_\theta^a \epsilon^\epsilon = 0. \)

3. Infinitesimal variations of submanifolds

We now consider an infinitesimal variation

(3.1) \[ \bar{x}^h = x^h + \xi^h(\gamma) \epsilon \]

of a submanifold \( M^{2n} \) of a Kaehlerian manifold \( M^{2m} \), where \( \xi^h(\gamma) \) is a vector
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Field of $M^{2m}$ defined along $M^n$ and $\varepsilon$ is an infinitesimal. We then have

$$(3.2) \quad \overline{B}_b^h = B_b^h + (\partial_b \xi^h) \varepsilon,$$

where $\overline{B}_b^h = \partial_b \xi^h$ are $n$ linearly independent vectors tangent to the varied submanifold. We displace $\overline{B}_b^h$ parallelly from the varied point $(\overline{x}^h)$ to the original point $(x^h)$. We then obtain

$$(3.3) \quad \overline{B}_b^h = B_b^h + (V_b \xi^h) \varepsilon,$$

neglecting the terms of order higher than one with respect to $\varepsilon$. In the sequel, we always neglect terms of order higher than one with respect to $\varepsilon$. Thus if we put

$$(3.4) \quad \delta B_b^h = \overline{B}_b^h - B_b^h,$$

we have

$$(3.5) \quad \delta B_b^h = (V_b \xi^h) \varepsilon.$$

If we put

$$(3.6) \quad \xi^h = \xi^a B_a^h + \xi^x C_x^h,$$

we find

$$(3.7) \quad V_b \xi^h = (V_b \xi^a - h_b^a \xi^x) B_a^h + (V_b \xi^x + h_b^a \xi^a) C_x^h.$$

When the tangent space of the varied submanifold at the varied point $(\overline{x}^h)$ is parallel to the tangent space of the original submanifold at the original point $(x^h)$, the infinitesimal variation (3.1) is said to be parallel.

From (3.5) and (3.7) we have

**Proposition 3.1.** A necessary and sufficient condition for the infinitesimal variation (3.1) to be parallel is

$$(3.8) \quad V_b \xi^h + h_b^a \xi^a = 0.$$

We next consider infinitesimal variations of the unit normals $C_y^h$. We denote by $\overline{C}_y^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and by $\overline{C}_y^h$ the vectors obtained from $\overline{C}_y^h$ by the parallel displacement of $\overline{C}_y^h$ from the point $(\overline{x}^h)$ to $(x^h)$. Then we have

$$(3.9) \quad \overline{C}_y^h = C_y^h + \Gamma_{ji}^h (x + \xi^x) \xi^j \overline{C}_y^i \varepsilon,$$

where $\Gamma_{ji}^h$ are Christoffel symbols fromed with $g_{ji}$.

We put

$$(3.10) \quad \delta C_y^h = \overline{C}_y^h - C_y^h$$

and assume that $\delta C_y^h$ is of the form

$$(3.11) \quad \delta C_y^h = (\eta_y^a B_a^h + \eta_y^x C_x^h) \varepsilon.$$

Then we have from (3.9), (3.10) and (3.11),

$$(3.12) \quad \overline{C}_y^h = C_y^h - \Gamma_{ji}^h \xi^j C_y^i \varepsilon + (\eta_y^a B_a^h + \eta_y^x C_x^h).$$
Now, applying the operator \( \delta \) to \( g_{ji} B_j^i C_j^i = 0 \) and using \( \delta g_{ji} = 0 \), \((3.5)\) and \((3.11)\), we find \((\nabla \phi^i + h_{ba} \xi^a = 0) \) and \( \eta_{by} = 0 \), where \( \xi^a = g_{yx} \xi^y \) and \( \eta_{by} = \eta^y g_{cb} \), from which

\[
(3.13) \quad \eta^y = - (\nabla^a \xi^y + h_{ae} \xi^e) \epsilon,
\]

where \( \nabla = g^{ae} \nabla_e \) Also, applying \( \delta \) to \( g_{ji} C_j^i C_j^i = g_{xy} - \delta_{xy} \), we find

\[
(3.14) \quad \eta_{yx} = \eta_{xy} = 0,
\]

where \( \eta_{yx} = \eta^y g_{xx} \).

We now compute the variations of \( f_a \xi^a, f_y \xi^y, f_s \xi^s \) and \( f_x \xi^x \) appearing in \((1.10)\) and \((1.11)\). First of all, we put

\[
F^h(x + \xi \epsilon) \bar{B}_j^i = (f_g^a + \delta f_g^a) \bar{B}_a^i - (f_s^a + \delta f_s^a) \bar{C}_x^i.
\]

Substituting \((3.2)\) and \((3.12)\) in this equation, using \( \nabla_x F^h = 0 \) and comparing the tangential and normal parts, we obtain

\[
(3.15) \quad \delta f_a^x = \left[ (\nabla \phi^a - h_{ae} \xi^e) f_a^x - f_s^a (\nabla \phi^a - h_{ae} \xi^e) \right] \epsilon + \left[ (\nabla \phi^a + h_{ae} \xi^e) f_a^x - f_s^a (\nabla \phi^a + h_{ae} \xi^e) \right] \epsilon
\]

and

\[
(3.16) \quad \delta f_y^x = \left[ f_a^x (\nabla \phi^a + h_{ae} \xi^e) + (\nabla \phi^a - h_{ae} \xi^e) f_x^a \right] \epsilon - \left[ (\nabla \phi^a + h_{ae} \xi^e) f_x^a - f_s^a (\nabla \phi^a + h_{ae} \xi^e) \right] \epsilon.
\]

We next put

\[
F^h(x + \xi \epsilon) \bar{C}_x^i = (f_y^a + \delta f_y^a) \bar{B}_a^i + (f_s^a + \delta f_s^a) \bar{C}_s^i.
\]

Then by a similar computation as above, we find

\[
(3.17) \quad \delta f_a = \left[ - f_s^a (\nabla \phi^a - h_{ae} \xi^e) - (\nabla \phi^a + h_{ae} \xi^e) f_a^x \right] \epsilon + \eta_s^a f_x^a + f_x^a (\nabla \phi^a + h_{ae} \xi^e) \epsilon
\]

and

\[
(3.18) \quad \delta f_y^x = \left[ - f_s^a (\nabla \phi^a + h_{ae} \xi^e) + (\nabla \phi^a + h_{ae} \xi^e) f_x^a \right] \epsilon + \eta_s^a f_x^a - f_x^a \eta_x^x \epsilon.
\]

### 4. Infinitesimal variations of invariant, anti-invariant, generic and CR submanifolds

Suppose that the submanifold \( M^a \) is invariant. Then we have \( f_b^x = 0 \) and \((3.16)\) becomes

\[
(4.1) \quad \delta f_y^x = \left[ f_y^x (\nabla \phi^x + h_{ad} \xi^d) - (\nabla \phi^x + h_{ad} \xi^d) f_y^x \right] \epsilon.
\]

An infinitesimal variation which carries an invariant submanifold into an invariant one is said to be invariant. From \((4.1)\), we have

**Proposition 4.1.** A necessary and sufficient condition for an infinitesimal variation \((3.1)\) to be invariant is that

\[
(4.2) \quad f_b^x (\nabla \phi^x + h_{ad} \xi^d) = (\nabla \phi^x + h_{ad} \xi^d) f_y^x.
\]
From Propositions 3.1 and 4.1 we have

**PROPOSITION 4.2.** A parallel variation is an invariant variation.

Suppose that the submanifold $M^n$ is anti-invariant. Then we have $f_\sigma = 0$ and (3.15) becomes

\[
\delta f_\sigma = \left[ (V^a \xi^e + h_{ae}^{\xi^e})f_x^\sigma - f_\sigma (V^a \xi^e + h_{ae}^{\xi^e}) \right] \epsilon.
\]

An infinitesimal variation which carries an anti-invariant submanifold into an anti-invariant one is said to be anti-invariant. From (4.3), we have

**PROPOSITION 4.3.** A necessary and sufficient condition for an infinitesimal variation (3.1) to be anti-invariant is that

\[
(4.4) \quad \left( V^a \xi^e + h_{ae}^{\xi^e} \right) f_x^\sigma = f_\sigma \left( V^a \xi^e + h_{ae}^{\xi^e} \right).
\]

From Propositions 3.1 and 4.3, we have

**PROPOSITION 4.4.** A parallel variation is an anti-invariant variation.

Suppose now that the submanifold $M^n$ is generic. Then we have $f_y = 0$ and (3.18) becomes

\[
\delta f_y = \left[ - f_y \left( V_\xi \xi^e + h_{ae}^{\xi^e} \right) + (V^a \xi^e + h_{ae}^{\xi^e}) f_e^\xi \right] \epsilon.
\]

An infinitesimal variation which carries a generic submanifold into a generic one is said to be generic. From (4.5), we have

**PROPOSITION 4.5.** A necessary and sufficient condition for an infinitesimal variation (3.1) to be generic is that

\[
(4.6) \quad f_y \left( V_\xi \xi^e + h_{ae}^{\xi^e} \right) = (V^a \xi^e + h_{ae}^{\xi^e}) f_e^\xi.
\]

From Propositions 3.1 and 4.5, we have

**PROPOSITION 4.6.** A parallel variation is generic.

Finally suppose that the submanifold $M^n$ is a CR submanifold. Then we have $f_y f_e^\xi = 0$. Substituting (3.15) and (3.16) into

\[
\delta (f_y f_e^\xi) = (\delta f_y) f_e^\xi + f_y (\delta f_e^\xi),
\]

we find

\[
\delta (f_y f_e^\xi) = \left[ (V^a \xi^e + h_{ae}^{\xi^e}) f_y f_e^\xi - f_y \left( V^a \xi^e + h_{ae}^{\xi^e} \right) f_e^\xi + f_y (V_\xi \xi^e + h_{ae}^{\xi^e}) f_e^\xi - f_y (V_\xi \xi^e + h_{ae}^{\xi^e}) f_e^\xi \right] \epsilon,
\]

from which, using (1.13) and (1.16),

\[
\delta (f_y f_e^\xi) = f_y \left[ (V^a \xi^e + h_{ae}^{\xi^e}) - (V^a \xi^e + h_{ae}^{\xi^e}) \right] f_e^\xi \epsilon
\]

An infinitesimal variation which carries a CR submanifold into a CR sub-
manifold is called a **CR variation**. From (4.7), we have

**Proposition 4.7.** A necessary and sufficient condition for an infinitesimal variation (3.1) to be a CR variation is that

\[(4.8)\]

\[f(y)\left(\nabla \xi_x + h_{xy} \xi_x d\right) - \left(\nabla \xi_y + h_{y} \xi_x d\right)f_x x = 0.\]

From Propositions 3.1 and 4.7, we have

**Proposition 4.8.** A parallel variation is a CR variation.

**References**


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