This paper is based on an idea of P. A. Griffiths [3] and M. H. Kwack [4]. The primary aim is to give a characterization of hyperbolic manifolds in terms of Schottky–Landau property. In addition, the author gives an elementary proof of R. Brody’s result (cf. [2]) as an application of this characterization.

1. Preliminaries

Let $M$ be a complex manifold of dimension $n$ and $T(M)$ its tangent bundle. A differential pseudometric is an upper semicontinuous function $F_M: T(M) \rightarrow \mathbb{R}$ satisfying

1. $F_M(z, v) \geq 0$, for any $(z, v) \in T(M)$, and
2. $F_M(z, rv) = r F_M(z, v)$ for any $r \in \mathbb{C}$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real numbers and of complex numbers, respectively. The integrated form of $F_M$ is given by, for all $x, y \in M$,

$$d_F(x, y) = \inf \int F_M(z, dz) = \inf \int_0^1 F_M(\gamma(t), \gamma'(t)) dt,$$

where the infimum is taken over all piecewise $C^1$ curves joining $x$ and $y$ in $M$. A well-known example of a differential pseudometric is a Kobayashi pseudometric which is defined by

$$K_M(z, v) = \{ |t| : f \in H(D, M), f(0) = z, f'(0)t = v \}$$

where $H(D, M)$ is the set of holomorphic mappings of the unit disc $D$ in the complex plane $\mathbb{C}$ into $M$. The upper semicontinuity of $K_M$ is proved by H. L. Royden (cf. [5]). A complex manifold is $F_M$-hyperbolic if each point in $M$ has a neighborhood $U$ and admits a positive number $m_U$ depending only on $U$ satisfying:

$$F_M(z, v) \geq m_U|v|,$$

for all $z \in U$ and $v \in T_z(M)$. $M$ is said to be hyperbolic if it is $K_M$-hyper-
bolic.

Given a pair of points \((z, w)\) in \(M\), we choose points \(z = z_0, z_1, \ldots, z_k = w\) of \(M\), points \(a_1, a_2, \ldots, a_k; b_1, \ldots, b_k\) of \(D\) and holomorphic mappings \(f_1, \ldots, f_k\) of \(D\) into \(M\) such that \(f_i(a_i) = z_{i-1}\) and \(f_i(b_i) = z_i\). The Kobayashi pseudometric is defined by

\[
d_M(z, w) = \inf \sum_{i=1}^{k} \rho_D(a_i, b_i),
\]

where the infimum is taken over all possible choices of points and functions as above and where \(\rho_D\) is the Poincaré metric on the unit disc. The following theorems show the relationships between the above terminologies.

**Theorem 1.1.** (H. L. Royden). \(d_M\) is the integrated form of \(K_M\).

**Theorem 1.2.** (H. L. Royden, T. J. Barth). For a complex manifold \(M\), the following are equivalent:

1. \(M\) is hyperbolic.
2. \(d_M\) is a proper distance.
3. \(d_M\) induces the standard topology on \(M\).

2. Lemmas

We shall denote by \(D(0, r)\) the open disc about origin with radius \(r\) in the complex plane. The following two lemmas generalize the lemmas given by M. H. Kwack [4].

**Lemma 2.1.** Let \(\{f_k\}\) be a sequence of holomorphic mappings from \(D = D(0, 1)\) into a hyperbolic manifold \(M\) which converges uniformly to an \(f\) on each compact subset of \(D\). Further assume that \(f_k(0) \in W\) for all indices \(k\) and for a fixed open and relatively compact subset \(W\) contained in a coordinate neighborhood \(U\) in \(M\). Then \(df_k(0)\) converges to \(df(0)\).

**Proof.** By Theorem 1.2., there exist positive numbers \(r_0, r_1\) and \(r_2\) such that

1. \(0 < r_0 < r_0 + r_1 < r_2\), and
2. \(\{p \in M : d_M(W, p) < r_2\} \subset U\).

Let \(t_0\) with \(0 < t_0 < 1\) satisfy that

\[
f(D(0, t_0)) \subset \{p \in M; d_M(W, p) < r_0\}.
\]

Choose a positive integer \(K\) such that for any \(k > K\) and for all \(x \in D(0, t_0)\),

\[
d_M(W, f_k(x)) \leq d_M(W, f(x)) + d_M(f_k(x), f(x)) \leq r_0 + r_1 < r_2,
\]

and hence \(f_k(D(0, t_0)) \subset U\). So the result follows from the Weierstrass’ theorem.
LEMMA 2.2. Let \( \{f_k\} \) be a sequence of holomorphic mappings of \( D \) into a hyperbolic manifold \( M \) such that \( f_k(0) \in W \) for all \( k=1,2,\ldots \) and for a fixed relatively compact open subset \( W \) of a relatively compact coordinate neighborhood \( U \) in \( M \).

Then, there is a positive real number \( t<1 \) such that on \( D(0,t) \) a subsequence of \( \{f_k\} \) converges uniformly to a holomorphic mapping \( f:D(0,t)\to M \).

**Proof.** Note that \( M \) is locally compact Hausdorff space. Let \( \varphi=(w_1,\ldots,w_n) \) be a coordinate function on \( U \). Since \( \bar{W} \cap U^c=\emptyset \) (empty set), where \( U^c=M-U \), choose a positive number \( t' \) such that \( Q=\{p\in M: d_M(W,p)<t'\} \) becomes a relatively compact subset of \( U \) by virtue of Theorem 1.2.

Since all the \( f_k \) are distance decreasing with respect to \( d_M \), we have
\[
 f_k(\{p\in D: \rho_D(0,p)<t'\}) \subseteq Q.
\]

Note that \( \varphi(Q) \) is a bounded subset of \( \mathbb{C}^n \). Thus by the Montel's theorem, we can choose \( t\leq t' \) such that a subsequence of \( \{f_k\} \) converging uniformly on \( D(0,t) \) to a holomorphic mapping \( f:D(0,t)\to M \).

3. Main theorem

**Theorem 3.1.** A complex manifold \( M \) is hyperbolic if and only if it has the Schottky-Landau property, i.e., given a point \( z_0\in M \) and a constant \( a>0 \), there is a neighborhood \( W \) of \( z_0 \) in \( M \) satisfying the following property: for any holomorphic mapping \( f:D(0,r)\to M \) with \( f(0)\in W \) and \( \|df(0)\|\geq a \), there is a constant \( R=R(a,W)>0 \) depending only on \( a \) and \( W \) such that \( r\leq R \).

**Proof.** Assume that \( M \) is hyperbolic. Let \( z_0\in M, a>0 \) be given. Choose a relatively compact coordinate neighborhood \( U \) about \( z_0 \) and a relatively compact open subset \( W \) of \( U \) containing \( z_0 \). Then we claim that this \( W \) does the job. Suppose that \( M \) does not have a Schottky-Landau property, then there is a sequence \( \{r_k\} \) of positive real numbers tending to infinity and a sequence of holomorphic mappings \( f_k:D(0,r_k)\to M \) with \( f_k(0)\in W \) and \( \|df_k(0)\|\geq a \). Let \( h_k(z)=f_k(r_kz) \), then \( h_k:D\to M \) satisfies the properties: \( h_k(0)\in W, \ h_k\in H(D,M) \) and \( \|dh_k(0)\|\geq ar_k \). Therefore, by Lemma 2.1, \( \{h_k\} \) cannot have any subsequence converging uniformly on a neighborhood of 0. But this contradicts Lemma 2.2.

Conversely, assume that \( M \) has the Schottky-Landau property. Let \( z_0\in M \) be given and let \( a=1 \). Then a neighborhood \( W \) of \( z_0 \) in \( M \) which forms the Schottky-Landau property. We claim first that, given \( (z,v)\in T(M) \), for every \( f\in H(D,M) \) with \( f(0)=z\in W \) and \( df(0)\eta=v \), for some \( \eta\in C \),
\[
 \|df(0)\|\leq \delta_W,
\]
for some $\delta_W$ depending only on $M$ and $W$. Assume the contrary, then for an arbitrarily given $B>0$, there is a holomorphic mapping $f:D\to M$ such that $f(0)=z\in W$, $df(0)\eta=v$ and $\|df(0)\|>B$. Define $h:D(0,B)\to M$ by $h(z)=f(z/B)$. Then $h$ is holomorphic and $\|dh(0)\|=(1/B)\|df(0)\|>1$.
Since $B$ can be arbitrarily large, this contradicts the Schottky-Landau property. Therefore there is a constant $\delta_W$ satisfying (*).

Hence, given $(z,v)\in T(M)$, $z\in W$, for any $\eta\in C$ such that there is an $f\in H(D,M)$ satisfying that $f(0)=z$ and $df(0)\eta=v$, we have

$$\delta_W |\eta| \geq \|df(0)\| |\eta| \geq |v|$$

or

$$|\eta| \geq m_W \|v\|,$$
where $m_W=1/\delta_W$. This completes the proof.

4. Application to compact cases

Using the device of Theorem 3.1., we will give an elementary proof of the following

**Theorem 4.1. (R. Brody).** Let $M$ be a compact complex manifold. Then the following are equivalent:

1. $M$ is hyperbolic.
2. $\sup \{\|df(0)\| : f\in H(D,M)\} < \infty$.
3. $M$ admits no complex line, i.e., there is no nonconstant holomorphic mapping from the complex plane into $M$.

**Proof.** (1) $\Rightarrow$ (2). By Theorem 3.1., for any $z_0\in M$, there is a neighborhood $W$ of $z_0$ in $M$ such that, for any $f\in H(D(0,r),M)$ with $f(0)\in W$ and $\|df(0)\| \geq 1$, there is a constant $R=R(W,1)$, depending only on $W$, such that $r\leq R$. It follows that every $f\in H(D,M)$ with $f(0)\in W$ satisfies $\|df(0)\| \leq R$. Since $M$ is compact, (2) holds.

(2) $\Rightarrow$ (1). Suppose that $M$ is not hyperbolic. Then by Theorem 3.1., there is a point $z_0\in M$ such that there is a sequence of holomorphic mappings $f_k:D(0,r_k)\to M$ satisfying $f_k(0)\in W$ for any coordinate neighborhood $W$ of $z_0$ and $\|df_k(0)\| \geq 1$, where $\{r_k\}$ is a sequence of positive real numbers tending to infinity. Let $g_k(z)=f_k(r_k z)$, then $g_k\in H(D,M)$ and $\|dg_k(0)\| \geq r_k$ so that $\sup \{\|df(0)\| : f\in H(D,M)\} = \infty$.

(1) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Suppose that $M$ is not hyperbolic. Then as above, there exists a sequence of holomorphic mappings $f_k:D(0,r_k)\to M$ such that $f_k(0)\in W$ and $\|df_k(0)\| \geq 1$ for a fixed relatively compact open subset $W$ of a relatively compact coordinate neighborhood in $M$, and such that $r_k$ tends to infinity.
as \( k \) becomes larger. Note that \( H(D(0, r_k), M) \) forms a normal family, since \( M \) is compact. Assume \( \{r_k\} \) is an increasing sequence with \( r_1 \geq 1 \).

Let \( \{f_{1,k}\} \) be a subsequence of \( \{f_k\} \) converging uniformly on all compact subsets of \( D(0, r_1) \). For each \( n=2, 3, 4, \ldots \), we choose recursively a sub-sequence \( \{f_{n,k}\} \) of \( \{f_{n-1,k}\} \) converging uniformly on all compact subsets of \( D(0, r_n) \).

Now choose the diagonal sequence \( \{f_{k,k}\} \) and let \( f \) be the limit function of the sequence of holomorphic mappings \( f_{k,k} \) restricted on the unit open disc. We claim that this \( f \) can be extended to a holomorphic mapping \( F:C \to M \), which will complete the proof.

Let \( \beta \) be an arbitrary point in \( C \). Then there is a natural number \( p \) such that \( |\beta| < r_p \). Let \( F_p \) be the limit function of the sequence

\[
\begin{align*}
&f_{p,p}|_{D(0,r_p)}, \, f_{p+1,p+1}|_{D(0,r_{p+1})}, \ldots.
\end{align*}
\]

Of course, \( F_p \) is holomorphic on \( D(0, r_p) \), especially at \( \beta \), and \( F_p|_D = f \).

Thus by the Monodromy theorem we get the result.

References


Seoul National University