SUBMANIFOLDS OF CODIMENSION 3 OF A KAHLERIAN MANIFOLD (I)

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0. Introduction

It is well known that a submanifold of codimension 3 of an Hermitian manifold admits an \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure induced from the almost Hermitian structure of the ambient manifold.

In the present paper we investigate a submanifold of codimension 3 of a \((2n+4)\)-dimensional Kaehlerian manifold admitting an \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure.

Firstly, we study the structure induced on the submanifold of codimension 3 of a \((2n+4)\)-dimensional Kaehlerian manifold. In section 1, we define the \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure and we show that this kind of structure gives an almost contact metric structure when \(\lambda^2 + \mu^2 + \nu^2 = 1\) and we find a necessary and sufficient condition that the \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure be antinormal. In section 2, we study some equations concerning the \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure and we show that in order for the structure to be antinormal, it is necessary and sufficient that \(h\) and \(f\) anticommute, where \(h\) is the second fundamental tensor with respect to the distinguished normal.

Next, we study the submanifold of codimension 3 of a \((2n+4)\)-dimensional Kaehlerian manifold of constant holomorphic sectional curvature \(c\). In section 3, we investigate the submanifolds satisfying the condition \(\lambda^2 + \mu^2 + \nu^2 = 1\) and we show that an umbilical submanifold with respect to the distinguished normal is an intersection of a complex cone and a sphere, that is, such a submanifold is an extended Brieskorn manifold. In section 4, we show that an antinormal minimal submanifold is a submanifold of a \((2n+3)\)-dimensional Euclidean space under some conditions. Moreover in this section, we show that a complete submanifold of codimension 3 of a Euclidean space \(E^{2n+4}\) is a plane or a ruled surface under some conditions. In section 5, we find a necessary and sufficient condition that the connection induced in the normal bundle of the submanifold to be trivial. Moreover in this section, we study a complete submanifold of codimension 3 of a \((2n+4)\)-dimen.

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sional Euclidean space $E^{2n+4}$ whose normal connection is flat and characterize this submanifold under some conditions.

1. Structures induced on submanifolds of codimension 3 of an almost Hermitian manifold

Let $M^{2n+4}$ be a $(2n+4)$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{U; x^A\}$ and denote by $g_{CB}$ components of the Hermitian metric tensor and by $F^A_B$ those of the almost complex structure tensor of $M^{2n+4}$, where here and in the sequel the indices $A, B, C, ...$ run over the range $1', 2', ..., (2n+4)'$. Then we have

\[(1.1) \quad F^C_B F_B^A = -\delta^C_A, \quad g_{ED} F^E_C F_B^D = g_{CB},\]

$\delta^C_A$ being the Kronecker delta.

Let $M^{2n+1}$ be a $(2n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhood $\{V; y^h\}$ and immersed isometrically in $M^{2n+4}$ by the immersion $i: M^{2n+1} \rightarrow M^{2n+4}$, where here and in the sequel the indices $h, i, j, ...$ run over the range $1, 2, ..., (2n+1)$. We identify $i(M^{2n+1})$ with $M^{2n+1}$ itself and represent the immersion by

\[(1.2) \quad x^A = x^A(y^h).\]

We now put $B_i^A = \partial_i x^A$, ($\partial_i = \partial / \partial y^i$). Then $B_i^A$ are $2n+1$ linearly independent vectors of $M^{2n+4}$ tangent to $M^{2n+1}$. And denote by $C^A, D^A,$ and $E^A$ three mutually orthogonal unit normals to $M^{2n+1}$. Then denoting by $g_{ij}$ components of the induced metric tensor of $M^{2n+1}$, we have

\[(1.3) \quad g_{ij} = g_{CD} B_i^C B_j^D\]

since the immersion is isometric.

As to the transforms of $B_i^A, C^A, D^A,$ and $E^A$ by $F_B^A$, we have respectively the following equations of the form

\[
(1.4) \quad F^C_A B_i^C = f_i^k B_k^A + u_i C^A + v_i D^A + w_i E^A,
\]

\[
(1.5) \quad F^A_B B^B = -u^h B_h^A - \nu D^A + \mu E^A,
\]

\[
(1.6) \quad F^A_B D^B = -v^h B_h^A + \nu C^A - \lambda E^A,
\]

\[
(1.7) \quad F^A_B E^B = -w^h B_h^A - \mu C^A + \lambda D^A,
\]

where $f_i^h$ is a tensor field of type $(1, 1)$, $u_i, v_i, w_i$ 1-forms and $\lambda, \mu, \nu$ functions in $M^{2n+1}$, $u^h, v^h,$ and $w^h$ being vector fields associated with $u_i, v_i$ and $w_i$ respectively.

Applying the operator $F$ to both sides of $(1.4)\sim(1.7)$, using $(1.1)$ and those equations and comparing tangential parts and normal parts of both sides, we find

\[(1.8) \quad f_i f_i^h = -\delta^h_i + u_i u^h + v_i v^h + w_i w^h,\]
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\[ f^h_i u^i = \nu v^h - \mu w^h, \]
\[ f^h_i v^i = -\nu u^h + \lambda w^h, \]
\[ f^h_i w^i = \mu u^h - \lambda v^h, \]
\[
\begin{align*}
&u_i u^i = 1 - \mu^2 - \nu^2, \quad u_i v^i = \lambda \mu, \\
v_i v^i = 1 - \nu^2 - \lambda^2, \quad v_i w^i = \mu \nu, \\
w_i w^i = 1 - \lambda^2 - \mu^2, \quad u_i w^i = \lambda \nu.
\end{align*}
\]

Also, from (1.1), (1.3) and (1.4), we find
\[
\begin{align*}
g_{ij} f^h_j f^h_i &= g_{ji} - u_i u_j - v_j v_i - w_j w_i. \\
\text{If we put } f_{ji} &= f^h_j g_{ii}, \text{ then we easily see that } f_{ji} = -f_{ij}.
\end{align*}
\]

Thus (1.8)~(1.11) show that the aggregate \((f^h_i, g_{ii}, u_i, v_i, w_i, \lambda, \mu, \nu)\) defines the so-called \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure on \(M^{2n+1}\) ([3], [6]).

An \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure is said to be antinormal if the tensor field \(S_{ji}^h\) of type (1,2) defined by
\[
S_{ji}^h = \beta[f, f]_{ji}^h + (\partial_j u_i - \partial_i u_j) u^h + (\partial_j v_i - \partial_i v_j) v^h + (\partial_j w_i - \partial_i w_j) w^h
\]
satisfies
\[
\partial_j u_i - \partial_i u_j = \nu (\partial_j v_i - \partial_i v_j) v^h + (\partial_j w_i - \partial_i w_j) w^h
\]
where \([f, f]_{ji}^h\) is the Nijenhuis tensor formed with \(f^h_i\), that is,
\[
\beta[f, f]_{ji}^h = f^h_j f^h_i - f^h_i f^h_j - (\partial_j f^h_i - \partial_i f^h_j) f^h_{ji}.
\]

We find from (1.9)
\[
f^h_i p^i = 0,
\]
where we have put
\[
p^h = \lambda u^h + \mu v^h + \nu w^h.
\]

From this and (1.10), we have
\[
u_i p^i = \lambda, \quad v_i p^i = \mu, \quad w_i p^i = \nu, \quad p_i p^i = \lambda^2 + \mu^2 + \nu^2.
\]

We now suppose that the aggregate \((f^h_i, g_{ii}, p^h)\) defines an almost contact metric structure. Then we get from the last equation of (1.16)
\[
\lambda^2 + \mu^2 + \nu^2 = 1
\]
because of \(p_i p^i = 1\). Conversely if the function \(\lambda, \mu, \nu\) satisfy (1.17), then (1.10) reduces to
\[
\begin{align*}
&u_i u^i = \lambda^2, \quad u_i v^i = \lambda \mu, \quad u_i w^i = \lambda \nu, \\
v_i v^i = \mu^2, \quad v_i w^i = \mu \nu, \quad w_i w^i = \nu^2.
\end{align*}
\]

Hence, it follows that
\[
\begin{align*}
&u_i = \lambda p_i, \quad v_i = \mu p_i, \quad w_i = \nu p_i
\end{align*}
\]
with the help of (1.16) and (1.18), where \(p_i = g_{ii} p^i\). Substituting (1.19) into (1.8) gives \(f^h_i f^h_i = -\delta^h_i + p_i p^h\) because of (1.17). Also substituting (1.19) into (1.11) and using (1.17), we find
Thus we see that the aggregate \((f^h, g^h, p^h)\) defines an almost contact metric structure. Concluding the developed above, we have

**Theorem 1.1.** ([6]) Let \(M^{2n+1}\) be a differentiable manifold with an \((f, g, u, v, \omega, \lambda, \mu, \nu)\)-structure. In order for the aggregate \((f, g, p)\), \(p\) being given by (1.15), to define an almost contact metric structure, it is necessary and sufficient that \(\lambda^2 + \mu^2 + \nu^2 = 1\).

In the sequel we suppose that the condition \(\lambda^2 + \mu^2 + \nu^2 = 1\) is satisfied on \(M^{2n+1}\). Suppose that the aggregate \((f, g, p)\) defines an almost contact metric structure and the induced structure is antinormal. Then we have (1.19) and consequently (1.13) reduces to

\[
[f, f]_i^k + (V_j p_i - V_i p_j) p^k = 2p_j (V_i p^k) - 2p_i (V_j p^k)
\]

with the help of (1.12) and (1.17). Thus we have

**Theorem 1.2.** Let \(M^{2n+1}\) be a differentiable manifold with an \((f, g, u, v, \omega, \lambda, \mu, \nu)\)-structure satisfying \(\lambda^2 + \mu^2 + \nu^2 = 1\). In order for this structure is antinormal, it is necessary and sufficient that (1.20) holds.

2. Structure equations of submanifolds of codimension 3 of a Kählerian manifold

Suppose that aggregate \((f, g, p)\) of \(f^h, g^h, p^h\) and \(p^h = \lambda^h + \mu v^h + \nu w^h\) defines an almost contact metric structure. Then we have (1.19) and consequently from (1.4)

\[
F_{C}^A B_i^C = f_i^h B_k^A + p_i N^A
\]

where \(N^A = \lambda C^A + \mu D^A + \nu E^A\) is an intrinsically defined unit normal to \(M^{2n+1}\) because \(C^A, D^A\) and \(E^A\) are mutually orthogonal unit normals to \(M^{2n+1}\) and \(\lambda^2 + \mu^2 + \nu^2 = 1\).

When a submanifold of an almost Hermitian manifold satisfies equation of the form (2.1), \(N^A\) being a unit normal to the submanifold, we say that the submanifold is semi-invariant with respect to \(N^A\) [1], [5]. We call \(N^A\) the distinguished normal to the semi-invariance. We take \(N^A\) as \(C^A\). Then we have \(\lambda = 1, \mu = 0, \nu = 0\) and consequently \(u^h = p^h, v^h = w^h = 0\) because of (1.10) and (1.15). Thus (1.4) \(\sim\) (1.7) becomes respectively

\[
(2.2) \quad F_{C}^A B_i^C = f_i^h B_k^A + p_i C^A,
\]

\[
(2.3) \quad F_{B}^A C^B = -f_i^h B_k^A,
\]

\[
(2.4) \quad F_{B}^A D^B = -E^A,
\]

\[
(2.5) \quad F_{B}^A E^B = D^A.
\]

Now denoting by \(V_j\) the operator of van der Waerden–Bortolotti covariant
differentiation with respect to $g_{ji}$, we have equations of Gauss for $M^{2n+1}$ of $M^{2n+4}$

\begin{align}
(2.6) \quad V_j B_i^A &= h_{ji} C^A + k_{ji} D^A + l_{ji} E^A,
\end{align}

where $h_{ji}, k_{ji}$ and $l_{ji}$ are the second fundamental tensors with respect to $C^A$, $D^A$ and $E^A$ respectively.

The equations of Weingarten are given by

\begin{align}
(2.7) \quad V_j C^A &= -h_{ji} B^A + l_{ji} D^A + m_j E^A, \\
(2.8) \quad V_j D^A &= -k_{ji} B^A + l_{ji} C^A + n_j E^A, \\
(2.9) \quad V_j E^A &= -l_{ji} B^A + m_j C^A - n_j D^A.
\end{align}

where $h_{ji} = h_{ji} g^{hi}, k_{ji} = k_{ji} g^{hi}, l_{ji} = l_{ji} g^{hi}, (g^{ji}) = (g_{ji})^{-1}, l_j, m_j$ and $n_j$ being the third fundamental tensors.

In the sequel we denote the normal components of $V_j C^A$ by $V_j C^A$. The normal vector field $C^A$ is said to be parallel in the normal bundle if we have $V_j C^A = 0$, i.e., $l_j$ and $m_j$ vanish identically.

We now assume that the ambient manifold $M^{2n+4}$ is Kaehlerian. Differentiating (2.2) covariantly along $M^{2n+1}$ and using (2.6) and (2.7), we easily find [6]

\begin{align}
(2.10) \quad V_j f^h &= -h_{ji} p^h + h_{ji} p^h, \\
(2.11) \quad V_j p^i &= -h_{ji} f^h, \\
(2.12) \quad k_{ji} &= -l_{ji} f^h - m_j p^i, \\
(2.13) \quad l_{ji} &= -k_{ji} f^h + l_j p^i,
\end{align}

from which

\begin{align}
(2.14) \quad k_{ji} p^i &= -m_j, \\
(2.15) \quad l_{ji} p^i &= l_j, \\
(2.16) \quad k &= -m_i p^i, \\
(2.17) \quad l &= l_i p^i,
\end{align}

where we have put $k = g^{hi} k_{ji}, l = g^{hi} l_{ji}$.

Transvecting (2.13) with $f^i$ and making use of (2.12), we obtain

\begin{align}
-k_{ij} - m_i p^h = k_{ij} f^h - f^h (l_i) p^i,
\end{align}

from which, taking the skew-symmetric part with respect to $i$ and $h$, $m_h p^i - m_i p^h = p_i (l_h f^i) - p_h (l_i f^i),$

or, transvecting with $p^h$ and using (2.16)

\begin{align}
(2.18) \quad l_i f^i = k p^i + m_i.
\end{align}

If we transvect (2.18) with $f^h$ and $l^i$ and take account of (2.17), we have respectively

\begin{align}
(2.19) \quad m_i f^i = l p^h - l_h, \\
(2.20) \quad k l + m_i l^i = 0.
\end{align}
Transvecting (2.12) with \( l^i_h \) and substituting (2.13), we find

\[ k_{ji} l^i_h = -(l_{ji} f^i + m_j p_i) (k_{hr} f^r + l_h p^r), \]

or, using (2.14) and (2.15)

\[ (2.21) \]

\[ k_{ji} l^i_h + k_{ij} l^j_i = -(l_{ji} m_i + l_{ij} m_j). \]

If we transvect (2.13) with \( l^i_h \) and substitute (2.12), we have

\[ (2.22) \]

\[ l_{ji} l^i_j + k_{ij} l^j_i = l_{ji} - m_j m_i \]

with the help of (2.14) and (2.15).

Now suppose that the \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure is antinormal, that is,

\[ f_j \nabla_i f^h - f^i \nabla_j f^h = \nabla_j f^i f^h + \nu (f_j p_i - f_i p_j) p^h = 2 p_j (\nabla_i p^h) - 2 p_i (\nabla_j p^h) \]

by virtue of (1.20). Substituting (2.10) and (2.11) into this, we find

\[ (f_j h^i_k + h^i_j f^h) p^h = -(f^i_j h^h_k + h^i_j f^h_k) p^h = 0 \]

and hence

\[ f^i_j h^i_k + h^i_j f^h_k = p^h q^i, \quad f^i_j h^h_k = 0, \]

for a certain vector field \( q^i \). From these equations we see that \( q^i = 0 \), and consequently

\[ (2.23) \]

\[ h_{ji} f^i_j = h_{ji} f^i_j. \]

Thus we have

**THEOREM 2.1.** Suppose that the \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure induced on a submanifold \( M^{2n+1} \) of codimension 3 of a Kählerian manifold \( M^{2n+4} \) satisfies \( \lambda^2 + \mu^2 + \nu^2 = 1 \). Then in order for this structure to be antinormal, it is necessary and sufficient that the second fundamental tensor \( h \) with respect to the distinguished normal and \( f \) anticommute.

The Gauss equations of \( M^{2n+1} \) for a Kählerian manifold \( M^{2n+4} \) are given by

\[ (2.24) \]

\[ K_{kij}^h = K_{DBA}^j B_k^D B_j^C B_i^B B_A^h + h_{jih} k_{ki} - h_{jik} k_{hi} \]

where \( B^h_A = g_{AC} g^i B_j^C \), \( K_{kij}^h \) and \( K_{DBA}^j \) being the Riemann–Christoffel curvature tensors of \( M^{2n+1} \) and \( M^{2n+4} \) respectively.

We now suppose that the ambient manifold is a Kählerian manifold \( M^{2n+4}(c) \) of constant holomorphic sectional curvature \( c \), that is, its curvature tensor has the form

\[ (2.25) \]

\[ K_{DBA}^j = \frac{c}{4} (\delta_D^A g_{CB} - \delta_C^A g_{DB} + F_D^A F_C B) \]

Substituting (2.25) into (2.24) and taking account of (1.3), (2.2) and (2.3), we have

\[ \ldots \]
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(2.26) \[ K_{ki}^h = \frac{c}{4} \left( \delta_k^h g_{ji} - \delta_j^h g_{ki} + f_j^h f_{ji} - f_j^h f_{ki} - 2 f_{kj} f_{ji}^h \right) \]
\[ + h_k^h h_{ji} - h_j^h h_{ki} + k_i^h k_j^h - k_j^h k_i^h - l_i^h l_{ji} - l_j^h l_{ki}. \]

In the same way by using (2.2)~(2.5), we can prove that equations of the Codazzi for \( M^{2n+4}(c) \) are given by

(2.27) \[ V_{ki}^h h_{ji} - V_{ji}^h k_{ki} + l_i^h k_{ji} - m_i^h l_{ji} + m_j^h l_{ki} = \frac{c}{4} \left( p_i^h f_{ji} - p_j^h f_{ki} - 2 p_i^h f_{kj} \right), \]
(2.28) \[ V_{ki}^h k_{ji} - V_{ji}^h k_{ki} + l_i^h k_{ji} - l_j^h k_{ki} - n_i^h l_{ji} + n_j^h l_{ki} = 0, \]
(2.29) \[ V_{ki}^h l_{ji} - V_{ji}^h l_{ki} + m_i^h k_{ji} - m_j^h k_{ki} + n_i^h k_{ji} - n_j^h k_{ki} = 0, \]
and those of the Ricci by

(2.30) \[ V_{ki}^h k_{ji} + h_k^h k_{ji} - h_j^h k_{ji} + m_k^h n_j - m_j^h n_k = 0, \]
(2.31) \[ V_{km}^j - V_{jm}^k + h_i^j l_{ji} - h_j^i l_{ki} + n_i^j l_{ji} - n_j^i l_{ki} = 0, \]
(2.32) \[ V_{kn}^j - V_{nj}^k + k_i^j l_{ji} - k_j^i l_{ki} + l_i^j m_j - l_j^i m_i = \frac{c}{2} f_{kj}. \]

3. Submanifolds of codimension 3 of \( M^{2n+4}(c) \) satisfying \( \lambda^2 + \mu^2 + \nu^2 = 1 \).

In this section we assume that the \((f, g, u, v, w, \lambda, \mu, \nu, \ldots)\)-structure induced on a submanifold \( M^{2n+1} \) of codimension 3 of a Kaehlerian manifold \( M^{2n+4}(c) \) of constant holomorphic sectional curvature \( c \) satisfies \( \lambda^2 + \mu^2 + \nu^2 = 1 \) and consequently the aggregate \((f, g, p)\) defines an almost contact metric structure.

We now suppose that the submanifold \( M^{2n+1} \) is umbilical with respect to the distinguished normal, that is, choosing \( CA \) as the distinguished normal,

(3.1) \[ h_{ji} = \tau g_{ji}, \quad k = 0, \quad l = 0 \]
for some function \( \tau \). Then (2.16), (2.17) and (2.20) imply that

(3.2) \[ l_i^h p^f = m_i^h p^f = l_i^m = 0 \]
and (2.11) becomes \( V_j p_i = \tau f_{ji} \), which shows that

\[ V_j V_j p_i = (V_j \tau) f_{ji} + \tau^2 (g_{hi} p_j - g_{ji} p_i) \]

with the help of (2.10) and the first relation of (3.1), from which, using the Ricci identity,

\[ - K_{ki}^h p_h = (V_j \tau) f_{ji} - (V_j \tau) f_{ki} + \tau^2 (g_{hi} p_j - g_{ji} p_i), \]
or, taking account of the first Bianchi identity,

(3.3) \[ (V_k \tau) f_{ji} + (V_j \tau) f_{ik} + (V_i \tau) f_{kj} = 0. \]

From this we can easily prove that \( \tau \) is a constant. Thus (2.27) reduces to

(3.4) \[ l_i^h k_{ji} + m_i^h l_{ji} - m_j^h l_{ki} = - \frac{c}{4} \left( p_i^h f_{ji} - p_j^h f_{ki} - 2 p_i^h f_{kj} \right). \]
because of (3.1). Transvecting (3.4) with \( p^i \) and using (2.14), (2.15) and (3.2), we get

\[(3.5) \quad m_j l_i - m_i l_j = \frac{c}{4} f_{ji}.\]

If we transvect (3.5) with \( f^ji \) and take account of (2.18), (2.19) and (3.1), then we get

\[(3.6) \quad m_i m^i = \frac{c}{4} n.\]

Also, transvecting (3.5) with \( m^i \) and using (3.2) and (2.19) with \( l=0 \), we find

\[\left( m_i m^i - \frac{c}{4} \right) l^i = 0,\]

or substituting (3.6) into this, \( c l^i = 0 \). Thus we have \( c = 0 \) because of (3.5).

From (2.10), (2.11) and (3.1), we have

\[(3.7) \quad V_j f^i = \tau (-g_{ji} p^h + \delta^h_{ji} p_i), \quad V_j p_i = \tau f_{ji}.\]

Hence, it follows that the aggregate \((f, g, p)\) defines a Sasakian structure if \( \tau \neq 0 \). We may consider \( \tau = 1 \) because \( \tau \) is a constant.

On the other hand, we see from (2.2) and (2.3) that the direct sum of the tangent space of \( M^{2n+1} \) and \( C^A \) is invariant. Then the ambient space being Euclidean, \( M^{2n+1} \) is an intersection of a complex cone with center at origin and with generator \( C^A \) and a \((2n+3)\)-dimensional sphere (See [6]). Thus we have

**THEOREM 3.1.** Let \( M^{2n+1} \) be a umbilical submanifold with respect to the distinguished normal \( C^A \) of a Kaehlerian manifold \( M^{2n+4}(c) \) of constant holomorphic sectional curvature \( c \) satisfying \( \lambda^2 + \mu^2 + \nu^2 = 1 \). Then \( M^{2n+1} \) is an intersection of a complex cone with generator \( C^A \) and a sphere.

We next prove the following

**THEOREM 3.2.** Let \( M^{2n+1} \) be a submanifold of codimension 3 of a Kaehlerian manifold \( M^{2n+1}(c) \) of constant holomorphic sectional curvature \( c \) with antinormal \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure satisfying \( \lambda^2 + \mu^2 + \nu^2 = 1 \). If the distinguished normal \( C^A \) is parallel in the normal bundle and the third fundamental tensor \( n_j \) satisfies

\[(3.8) \quad V_j n_i - V_i n_j = 2\alpha f_{ji}\]

for a certain function \( \alpha \), then \( M^{2n+1} \) is a hypersurface of \( M^{2n+2}(c) \).

**Proof.** Since \( V_j C^A = 0 \), that is, \( l_j \) and \( m_j \) vanish identically, we have from (2.32)

\[V_k n_j - V_j n_k + 2k^i l_{ij} = \frac{c}{2} f_{kj} \]
because of (2.21). Thus (3.8) reduces to

(3.9) \[ k_1^1 l_{ji} = \left( \frac{c}{4} - \alpha \right) f_{kj}. \]

Transvecting (3.9) with \( f_j^k \) and taking account of (2.13) with \( l_j = 0 \), we find

(3.10) \[ l_{ji} l_{jt} = \left( \alpha - \frac{c}{4} \right) \cdot (g_{ji} - p_j p_i). \]

Therefore, it follows that

(3.11) \[ k_j^i k_i^j = \left( \alpha - \frac{c}{4} \right) \cdot (g_{ji} - p_j p_i) \]

because of (2.22) with \( l_j = m_j = 0 \).

Since \( l_j = m_j = 0 \), (2.28), (2.29) and (2.31) reduces respectively to

(3.12) \[ V_{jk} l_{ji} - V_{j} l_{ki} = n_k l_{ji} - n_j l_{ki}, \]

(3.13) \[ V_{kl} l_{ji} - V_{j} l_{ki} = -n_k l_{ji} + n_j l_{ki}, \]

(3.14) \[ h^i_{jl} l_{ji} - h^i_{jl} l_{ki} = 0. \]

Transvecting (3.14) with \( l_i^k \) and making use of (3.10), we find

\[ \left( \alpha - \frac{c}{4} \right) \cdot (h_{ji} - p_j h_{jt} p_t) - h_{st} l_{lt} = 0, \]

from which, taking the skew-symmetric part,

(3.15) \[ \left( \alpha - \frac{c}{4} \right) \cdot (h_{jt} p^t - \beta p_j) = 0, \]

where we have put

(3.16) \[ \beta = h_{st} p^t p^s. \]

As in the proof of Theorem 3.1, we can easily from (3.8) see that \( \alpha \) is a constant by using the Ricci and Bianchi identities.

Differentiating (3.10) covariantly and using the fact that \( \alpha - \frac{c}{4} \) is a constant, we obtain

(3.17) \[ l_{j}^t (V_i l_{jt}) + l_{i}^t (V_i l_{ij}) = \left( \frac{c}{4} - \alpha \right) \cdot \{ (V_k p_j) p_i + (V_k p_i) p_j \}, \]

from which, taking the skew-symmetric part with respect to \( k \) and \( j \) and substituting (3.13),

\[ l_{j}^t (n_j k_{kt} - n_k k_{jt}) + l_{i}^t (V_i l_{kt} - n_k k_{it} + n_i k_{kt}) - l_{k}^t (V_i l_{jt} - n_j k_{it} + n_i k_{jt}) \]

\[ = \left( \frac{c}{4} - \alpha \right) \cdot \{ (V_k p_j - V_j p_k) p_i + (V_k p_i) p_j - (V_j p_i) p_k \}, \]

or using (2.21) with \( l_j = m_j = 0 \),

\[ l_{j}^t (V_i l_{jt}) - l_{k}^t (V_i l_{jt}) + 2 n_j l_{jt} k_{kt} \]

\[ = \left( \frac{c}{4} - \alpha \right) \cdot \{ (V_k p_j - V_j p_k) p_i + (V_k p_i) p_j - (V_j p_i) p_k \}. \]

Interchanging the indices \( k \) and \( i \), we get
Adding (3.17) to (3.18), we find
\[
\frac{2}{l_j^t} + 2n_k \ell_j^t \ell_k^t = \left( \frac{c}{4} - \alpha \right) \cdot \left\{ (V_k P_j - V_j P_k) P_i + (V_i P_k + V_k P_i) P_j + (V_i P_j - V_j P_i) P_k \right\},
\]
from which, transvecting \( P^i \) and taking account of (2.11), (2.15) with \( l_j = 0 \) and (3.15),
\[
\frac{c}{4} - \alpha = 0.
\]
Since the induced structure is antinormal, by transvecting \( f_j^k \) and taking account of (2.23) and (3.15), we find
\[
\frac{c}{4} - \alpha = 0.
\]
If \( \frac{c}{4} - \alpha \neq 0 \), then (2.11) becomes \( V_j P_i = 0 \) because of \( h_{ji} = \beta_j P_i \). Thus (2.27) reduces to
\[
(V_k \beta) P_j P_i - (V_j \beta) P_k P_i = \frac{c}{4} \left( p_k f_{ji} - p_j f_{ki} - 2 p_{ij} f_{kj} \right)
\]
because of \( l_j = n_j = 0 \).
Transvecting (3.21) with \( p^i p^i \), we obtain \( V_{ii} \beta = (p^i V_i \beta) p_k \). Hence (3.21) implies that \( c \) is zero. Consequently the ambient manifold is Euclidean. According to Lemma 5.4 of [6], \( \alpha \) must be zero. It contradicts the fact that \( \frac{c}{4} - \alpha \neq 0 \). Thus we have \( \alpha = \frac{c}{4} \). Thereby (2.26) \sim (2.32) become the structure equations for a hypersurface of \( M^{2n+2}(c) \). Thus we complete the proof of the theorem.

4. Antinormal submanifolds of codimension 3 of \( M^{2n+4}(c) \) satisfying \( \lambda^2 + \mu^2 + \nu^2 = 1 \).

In this section we assume that the induced \((f, g, u, v, w, \lambda, \mu, \nu)\)-structure induced on a submanifold \( M^{2n+1} \) of codimension 3 of a Kaehlerian manifold \( M^{2n+4}(c) \) of constant holomorphic sectional curvature \( c \geq 0 \) satisfies \( \lambda^2 + \mu^2 + \nu^2 = 1 \) and is antinormal. Then we have (2.23).

Transvecting (2.23) with \( p^i \) and taking account of (1.14), we get
\[
(h_{ji} P^i) f_j^i = 0,
\]
from which,
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(4.1) \[ h_{ij}^{t} = \beta p_{i} \]

because of (3.16).

Differentiating (4.1) covariantly and substituting (2.11), we find

\[ (\nabla_{j} h_{ij})^{t} - h^{t}_{ij} f_{i} = (\nabla_{j} \beta) p_{i} - \beta h_{ij} f_{i}, \]

from which, taking the skew-symmetric part and using (2.23) and (2.27)

\[ \left\{ l_{j} k_{it} - l_{j} k_{it} + m_{j} l_{it} - m_{i} l_{jt} + \frac{c}{4} (p_{j} f_{it} - p_{i} f_{jt} - 2 p_{i} f_{ji}) \right\} p^{t} \]

\[ = 2 h^{t}_{ij} h_{ij} f_{i}^{t} + (\nabla_{j} \beta) p_{j} - (\nabla_{i} \beta) p_{j}, \]

or, using (2.14), (2.15) and (2.23)

\[ (f^{T} j f_{j})^{t} = (f^{T} j f_{j})^{t} - (f^{T} j f_{j})_{i} + (f^{T} j f_{j})_{P_{i}} + \frac{1}{2} \{ (\nabla_{i} \beta) p_{j} - (\nabla_{j} \beta) p_{i} \} + l_{m} m_{j} - l_{m} m_{i}. \]

If we transverse (4.2) with \( p^{t} \), then we have

\[ \frac{1}{2} \nabla_{i} \beta = \frac{1}{2} (p^{T} \nabla_{i} \beta) p_{i} + l_{m} + k_{i} \]

because of (2.16) and (2.17). Thus (4.2) gives

\[ h^{t}_{ij} f_{j}^{t} + \frac{c}{4} f_{ji} = (m_{i} p_{j} - m_{j} p_{i}) + k (l_{i} p_{j} - l_{j} p_{i}) + l_{m} - l_{m} m_{i}. \]

Transvecting (4.4) with \( f_{k}^{t} \) and using (4.1), we find

\[ - h^{t}_{ij} h_{kl} + \beta^{2} p_{j} p_{k} + \frac{c}{4} (- e_{ik} + e_{ij} p_{k}) \]

\[ = (l_{i} - l_{j}) f_{k}^{t} m_{k} - (m_{i} - k p_{i}) f_{k}^{t} l_{i}, \]

from which, substituting (2.18) and (2.19),

\[ (4.5) \]

\[ h^{t}_{ij} h_{kl} - \beta^{2} p_{j} p_{k} + \frac{c}{4} (e_{ik} - p_{j} p_{k}) \]

\[ = l_{j} l_{k} + m_{j} m_{k} - l (l_{i} p_{j} + l_{j} p_{i}) + k (m_{i} p_{k} + m_{k} p_{i}) + (l^{2} + k^{2}) p_{i} p_{k}. \]

On the other hand, transvecting (2.23) with \( f^{j} i \) and making use of (4.1), we have

\[ (4.6) \]

\[ h = \beta, \]

where we have put \( g^{j} i h_{ji} = h. \)

Using this fact, (4.5) reduces to

\[ h^{t}_{ij} h^{t}_{kl} + \frac{c}{4} g_{ji} = \left( h^{2} + k^{2} + l^{2} + \frac{c}{4} \right) p_{j} p_{i} + l_{j} l_{i} + m_{j} m_{i} \]

\[ + k (m_{i} p_{j} + m_{j} p_{i}) - l (l_{i} p_{j} + l_{j} p_{i}), \]

which implies

\[ (4.7) \]

\[ h^{t}_{ij} h^{t}_{ji} = h^{2} - k^{2} - l^{2} + l_{j} l_{i} + m_{j} m_{i} - \frac{nc}{2} \]

with the help of (2.16) and (2.17). Since the left-hand side of (4.8)
becomes $||h_{ji} - h_{pj} p_i||^2$ because of (4.1) and (4.6), (4.8) can be written as

(4.9) $||h_{ji} - h_{pj} p_i||^2 = l_i l^i + m_i m^i - \left( k^2 + l^2 + \frac{nc}{2} \right)$.

For an eigenvalue $\rho$ of $h_j$ corresponding to the eigenvector orthogonal to $p^i, l^i$ and $m^i$, we have from (4.7) that $\rho^2 + \frac{c}{4} = 0$ if $n \geq 2$. Thus it follows that $\rho \leq 0$ because the eigenvalue is real and hence $\rho = 0$.

We now suppose that $V_j A^c = 0$ and $M^{2n+1}$ is minimal. Then we have from (4.8) with $\rho = 0$ that $h_{ji} = 0$. Therefore (2.26) $\sim$ (2.32) mean that $M^{2n+1}$ is a submanifold of codimension 2 in a Euclidean space $E^{2n+3}$ because of $c = 0$.

Hence we have

PROPOSITION 4.1. Let $M^{2n+1}$ $(n \geq 2)$ be a minimal submanifold of codimension 3 of a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature $c \geq 0$ such that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on $M^{2n+1}$ defines an almost contact metric structure $(f, g, \varphi)$, $\varphi$ being given by (1.15) and is anti-normal. If the distinguished normal $\varphi A$ is parallel in the normal bundle, then $M^{2n+1}$ is a submanifold of a Euclidean space $E^{2n+3}$.

Denoting by $K_{ji} = K_{iji}$ and $K = g^{ji} K_{ji}$ the Ricci tensor and the scalar curvature of $M^{2n+1}$ respectively, we then have from (2.26)

$$K_{ji} = \frac{c}{4} \{ (2n+3) g_{ji} - 3 p_j p_i \} + h h_{ji} + k k_{ji} + l l_{ji}$$

$$- h_{ji} h^i - k_{ji} k^i - l_{ji} l^i,$$

from which

$$K = n(n+2) \cdot c + h^2 + k^2 + l^2 - h_{ji} h^{ji} - k_{ji} k^{ji} - l_{ji} l^{ji},$$

or, substituting (4.8) and taking account of (2.22)

$$K = \frac{n(2n+5)}{2} c - 2( l^2 - l^2 ) - 2( k^2 k^{ji} - k^2 ),$$

which means

(4.10) $K = \frac{n(2n+5)}{2} c - ||l_j p_i - l_i p_j||^2 - 2||k_{ji} - k_{pj} p_i||^2$

with the help of (2.14) $\sim$ (2.17). Thus if $K \geq \frac{n(2n+5)}{2} c$ holds, we have $l_j p_i - l_i p_j = 0, k_{ji} = k_{pj} p_i$. Hence (2.12) and (2.13) imply that $l_{ji} = l_{pj} p_i, m_{ji} = m_{pj} p_i$. It follows from (4.9) that $||h_{ji} - h_{pj} p_i||^2 + \frac{n}{2} c = 0$ and consequently $h_{ji} = h_{pj} p_i$ and $c = 0$ because of $c \geq 0$. Thus (2.10) and (2.11) becomes $V_j f^k = 0$, $V_j p_i = 0$. And (2.26) reduces to $K_{ji} = 0$.

Therefore we have
PROPOSITION 4.2. Let $M^{2n+1}$ be a submanifold of codimension 3 of a Kählerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature $c \geq 0$ such that the $(f, g, u, v, w, \lambda, \mu, \nu)-$structure induced on $M^{2n+1}$ is antinormal and satisfies $\lambda^2 + \mu^2 + \nu^2 = 1$. If the scalar curvature $K$ of $M^{2n+1}$ satisfies $K \geq \frac{n(2n+5)}{2}c$ at every point, then $M^{2n+1}$ is a locally Euclidean space with the second fundamental tensors of the forms

$$h_{ji} = hp_jp_i, \quad k_{ji} = kp_jp_i, \quad l_{ji} = lp_jp_i,$$

and admits a cosymplectic structure.

We now prove the following

THEOREM 4.3. Let $M^{2n+1}$ be a complete submanifold of codimension 3 of a Euclidean space $E^{2n+4}$ with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)-$structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. If the distinguished normal $CA$ is parallel in the normal bundle and the third fundamental tensor of $M^{2n+1}$ satisfies

$$V_jn_i - V_in_j = 2xf_{ji},$$

then $M^{2n+1}$ is a plane or a ruled surface which is generated by parallel displacements of a plane $E^{2n}$ along a plane curve orthogonal to $E^{2n}$.

Proof. Since $V_jCA = 0$, that is, $l_j = m_j = 0$, we have from (4.9) with $c = 0$

$$h_{ji} = hp_jp_i.$$

From (4.11) we can prove that $\alpha = 0$ and hence

$$k_{ji} = 0, \quad l_{ji} = 0.$$

(See Lemma 5.4 and Theorem 5.5 of [6]). Thus (2.7) $\sim$ (2.9) reduce to respectively

$$V_jCA = -hp_j(\phi^bB^A_h), \quad V_jDA = n_jE^A, \quad V_jE^A = -n_jD^A$$

because of $l_j = m_j = 0$. Also (2.11) and (4.12) imply that

$$V_j\phi^h = 0.$$

Let $M'$ be a real hypersurface of $M^{2n+1}$ which is defined by the Pfaffian form $\omega = p_i dx^i$ and be covered by a system of coordinate neighborhoods $\{U'$ ; $\xi^a\}$, where the indices $a, b, c$ run over the range $1', 2', \ldots, 2n'$.

Let $i' : M' \to M^{2n+1}$ be an isometric immersion represented by $\phi^h = y^h(\xi^a)$. Putting $B^h_a = \partial_d y^h$, $(\partial_d = \partial/\partial \xi^a)$, then $B^h_a$ are $2n$ linearly independent vectors of $M^{2n+1}$ tangent to $M'$. By definition, $\phi^h$ is a unit normal to $M'$. Now we put

$$B^A_a = B^h_a B^A_h, \quad P^A = y^h B^A_h.$$

Then $P^A$ is a unit normal vector field orthogonal to $CA$, $DA$ and $E^A$. In this case, we can easily see that $M'$ is a totally geodesic submanifold of $E^{2n+4}$.
because of (4.12), (4.13) and (4.15). Consequently $M'$ is a plane $E^{2n}$ parallel along $p^h$ because the ambient space is Euclidean.

If we take account of (4.12) and (4.13), then (2.6) becomes $V_j B^h A = h p_j p_i C^A$, or by transvecting $p^h$

\[(4.17) \quad V_j P^A = h p_j C^A\]

with the help of (4.15).

From (4.14) and (4.17), we have

\[p^j V_j C^A = -h P^A, \quad p^j V_j P^A = h C^A,\]

which shows a plane curve with curvature $h$ on a complex two dimensional plane $C^2$ spaned by $\{P^A, C^A, D^A, E^A\}$. Then the orthogonal complementary space of $C^2$ is a plane $E^{2n}$. Hence $M^{2n+1}$ is a ruled surface which is generated by parallel displacements of $E^{2n}$ along a curve on $C^2$ if $h \neq 0$. If $h=0$, then $M^{2n+1}$ is a plane in $E^{2n+4}$ because of (4.12) and (4.13). This completes the proof the theorem.

Replacing the condition (4.11) in Theorem 4.3 by $K \geq 0$, we can see that

\[k_{ji} = 0, \quad l_{ji} = 0.\]

In fact, since $V_j^{-1} C^A = 0$, (4.9) with $c = 0$ implies that $h_{j i} = h p_j p_i$. Consequently (4.10) with $c = 0$ becomes $K = -2 k_{ji} k_{ji}$ with the help of $l_{j} = 0$ and $k = 0$. It follows that $k_{ji} = 0$ because of $K \geq 0$ and hence $l_{ji} = 0$ by virtue of (2.22).

According to Theorem 4.3, we have

**Corollary 4.4.** Let $M^{2n+1}$ be a complete submanifold of codimension 3 of a Euclidean space $E^{2n+4}$ whose normal connection is flat. Then we have

\[k_{ji} = m_j p_i = m_i p_j,\]

\[k_{ji} = k p_i p_j.\]
In the same way we have from (2.13), (2.15) and (2.17) that

\begin{equation}
(l_{ij} = l_{pj}p_{j}, \quad l_j = l_{pj}).
\end{equation}

Thus (4.9) with \( \epsilon = 0 \) implies

\begin{equation}
(h_{ij} = h_{pj}p_j)
\end{equation}

with the help of the fact that \( l_{ij} = l^2 = l^2 \) and \( m_{ij} = k^2 \).

Conversely, if (5.3) and (5.5) are satisfied, then we easily see that (2.23) and (5.1) are valid. Therefore we have

**PROPOSITION 5.1.** Suppose that the \((f, g, u, \nu, \omega, \lambda, \mu, \nu)\)-structure induced on a submanifold \(M^{2n+1}\) of codimension 3 of a Kählerian manifold \(E^{2n+4}\) satisfies \( \lambda^2 + \mu^2 + \nu^2 = 1 \) and consequently \((f, g, \nu)\) defines an almost contact metric structure. Then in order for these structures to be antinormal and the connection induced in the normal bundle of \(M^{2n+1}\) to be trivial, it is necessary and sufficient that the second fundamental tensors of \(M^{2n+1}\) have the form

\begin{equation}
(h_{ij} = h_{pj}p_j, \quad k_{ij} = k_{pj}p_j, \quad l_{ij} = l_{pj}p_j).
\end{equation}

On the other hand, the mean curvature vector \(H\) of \(M^{2n+1}\) is given by

\[
H = \frac{1}{2n+1} (hC + kD + lE).
\]

If we now take the distinguished normal as a direction of the mean curvature vector if \(H \neq 0\), that is, we choose the normals \(H/||H||\), \(D\) and \(E\) such that \(H = ||H||C\), then we have

\[
\begin{pmatrix}
'c' \\
'd' \\
'e'
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
C \\
D \\
E
\end{pmatrix}
\]

for some constant \(\theta\), where \(C = H/||H||\). This means

\begin{equation}
'c = c, \quad 'd = \cos \theta \cdot D + \sin \theta \cdot E, \quad 'e = -\sin \theta \cdot D + \cos \theta \cdot E.
\end{equation}

As to the transforms of \(B_i^A, C^A, D^A\) and \(E^A\) by \(F_B^A\), we have respectively the equations of the form

\[
F_B^A B_i^B = f_i^B B_h^A + \nu_i^C C^A + \nu_j^D D^A + \nu_i^E E^A,
\]

\[
F_B^A C^B = -\nu_i^B C^A + \nu_j^D D^A + \nu_i^E E^A,
\]

\[
F_B^A D^B = -\nu_i^B C^A - \nu_j^D D^A - \nu_i^E E^A,
\]

\[
F_B^A E^B = -\nu_i^B C^A + \nu_j^D D^A - \nu_i^E E^A.
\]

If we apply the operator \(F\) to these equations and use (5.7), we obtain

\begin{equation}
'\lambda = \lambda, \quad '\mu = \cos \theta \cdot \mu + \sin \theta \cdot \nu, \quad '\nu = -\sin \theta \cdot \mu + \cos \theta \cdot \nu,
\end{equation}

which shows that \(\lambda = 1, \quad '\mu = '\nu = 0\) if \(\lambda = 1, \quad \mu = \nu = 0\), that is, although the
normals $C, D$ and $E$ are rotated by the fixed angle $\theta$, we may take $H$ as distinguished normal.

Let $'h_{ji}, 'k_{ji}$ and $'l_{ji}$ be the second fundamental tensors with respect to $'C', 'D$ and $'E$, and $'l_j, 'm_j$ and $'n_j$ the third fundamental tensors corresponding to $l_j, m_j$ and $n_j$ respectively.

By differentiating $(5.7)$ covariantly and taking account of $(2.7) \sim (2.9)$, we then have

\begin{align*}
'h_{ji} &= 'h_{ji}, \quad 'k_{ji} = \cos \theta \cdot 'k_{ji} + \sin \theta \cdot 'l_{ji} \\
'l_{ji} &= -\sin \theta \cdot 'k_{ji} + \cos \theta \cdot 'l_{ji},
\end{align*}

(5.9)

\begin{align*}
'l_j &= \cos \theta \cdot 'l_j + \sin \theta \cdot 'm_j, \quad 'm_j = -\sin \theta \cdot 'l_j + \cos \theta \cdot 'm_j, \quad 'n_j = 'n_j
\end{align*}

(5.10)

because $\theta$ is a constant.

Since the distinguished normal as a direction of the mean curvature vector, we have

\begin{align*}
'h &= 'h, \quad 'k = 'k, \quad 'l = 'l = 0,
\end{align*}

(5.11)

where we have put $'h = 'h, 'k = 'k$, and $'l = 'l$.

By using (5.9), we can easily verify that (2.23) and (5.1) are of intrinsic characters. Hence (5.6) implies

\begin{align*}
'h_{ji} &= 'h_{ji}, 'l_{ji} = 'l_{ji} = 0
\end{align*}

because of (5.11). As in the proof of Theorem 4.3, $M^{2n+1}$ is a ruled surface which is generated by parallel displacements of a plane $E^{2n}$ along a plane curve orthogonal to $E^{2n}$ if $H \neq 0$. Thus we have

**THEOREM 5.2.** Let $M^{2n+1}$ be a complete submanifold of codimension 3 of a Euclidean space $E^{2n+4}$ with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)-structure$ satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$ whose normal connection is flat. If we take the distinguished normal as a direction of the mean curvature vector $H$, then $M^{2n+1}$ is a ruled surface which is generated by parallel displacements of a plane $E^{2n}$ along a plane curve orthogonal to $E^{2n}$ provided that $H \neq 0$. If $H = 0$, then $M^{2n+1}$ is a plane $E^{2n+1}$.

Combining Proposition 4.2 and Theorem 5.2, we have

**COROLLARY 5.3.** Let $M^{2n+1}$ be a complete submanifold of codimension 3 of a Euclidean space $E^{2n+4}$ with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)-structure$ satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. If we take the distinguished normal as a direction of the mean curvature vector and the scalar curvature of $M^{2n+1}$ is nonnegative at every point, we have the same conclusions of Theorem 5.2.
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References


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