A CERTAIN POLYNOMIAL STRUCTURE

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0. Introduction

K. Matsumoto[8] has introduced the pseudo-\( f \)-structure defined by a tensor field \( f \) of type \((1,1)\) satisfying \( f^3-f=0 \) and investigated the integrability conditions of the pseudo-\( f \)-structure. On the other hand, I. Sato [11] has studied an almost paracontact structure \((f, \xi, \eta)\) of the pseudo-\( f \)-structure of rank \( n-1 \). The purpose of the present paper is to introduce a pseudo-framed structure and to obtain the results analogous to the properties of a framed structure. In § 1 we introduce a pseudo-framed structure of rank \( r \) and give an example of a manifold with such a structure. This structure is a generalization of an almost product structure and almost paracontact structure.

In § 2 we study structures induced on a product manifold of two pseudo-framed manifolds and prove the manifold \( M \times \mathbb{R}^{n-r} \) has an almost product structure. In § 3 we define the normal pseudo-framed structure and prove that the product manifold of two normal pseudo-framed manifolds has a normal pseudo-framed structure.

1. Pseudo-framed structure

Let \( M \) be an \( n \)-dimensional differentiable manifold of class \( C^\infty \). If there exists a tensor field \( f \) of type \((1,1)\) of constant rank \( r \) satisfying the polynomial equation:

\[
f^3-f=0,
\]

then we call the structure a pseudo-\( f \)-structure of rank \( r \) and the manifold \( M \) pseudo-\( f \)-manifold of rank \( r \) ([8]). This structure is a generalization of an almost product structure \((r=n)\) and almost paracontact structure \((r=n-1)\) ([11]).

If we put

\[
s=f^2, \quad t=-f^2+I,
\]

where \( I \) is the identity transformation field, then we get

\[
s+t=I, \quad s^2=s, \quad t^2=t, \quad fs=f, \quad ft=0, \quad st=0.
\]

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The operators \( s \) and \( t \) acting in the tangent space at each point of \( M \) are therefore complementary projection operators and there exist complementary distributions \( S \) and \( T \) corresponding to the operators \( s \) and \( t \), respectively. Then the distribution \( S \) is \( r \)-dimensional and distribution \( T \) is \( (n-r) \)-dimensional.

Let \( M \) be a manifold with pseudo-\( f \)-structure of rank \( r \). There exist \( n-r \) vector fields \( \xi_x \) spanning the distribution \( T \) and its dual 1-forms \( \eta_x \), where the indices \( x, y, z \) run over the range \( \{1, 2, \ldots, n-r\} \). Then we can put
\[
t = \eta_x \otimes \xi_x, \quad \eta_x(\xi_y) = \delta_{xy},
\]
where \( \delta_{xy} \) is the Kronecker’s delta, the summation convention being employed here and in the sequel. Therefore, for any vector field \( X \) we have
\[
(1.4) \quad sX = f^2X, \quad tX = \eta_x(X)\xi_x,
\]
from which
\[
(1.5) \quad f^2 = I - \eta_x \otimes \xi_x.
\]
From (1.3) and (1.5) we easily see that
\[
(1.6) \quad f^2 = 0, \quad \eta_x \circ f = 0.
\]
If there exist on \( M \) vector fields \( \xi_x \) and 1-forms \( \eta_x \) satisfying (1.4), (1.6) and (1.7), then the set \( (f, \xi_x, \eta_x) \) is called a pseudo-\( f \)-structure with complementary frame, or simply, a pseudo-framed structure and the manifold \( M \) a pseudo-framed manifold.

Let \( M \) be a manifold with pseudo-framed structure of rank \( r \). Then there exists on \( M \) a Riemannian metric \( g \) such that
\[
(1.7) \quad g(X, \xi_x) = \eta_x(X),
\]
\[
(1.8) \quad g(fX, fY) = g(X, Y) - \eta_x(X)\eta_x(Y),
\]
for any vector fields \( X \) and \( Y \) on \( M \).

If we put
\[
(1.9) \quad F(X, Y) = g(X, fY),
\]
then we get
\[
(1.10) \quad F(X, Y) = F(Y, X),
\]
which shows that \( F \) is a symmetric tensor.

Now, as an example, we consider a submanifold \( N \) of codimension \( r \) of an \( n \)-dimensional almost product manifold \( M \) with structure tensor \( (J, G) \). If \( B \) denotes the differential of imbedding \( i : N \rightarrow M \) and \( X \) and \( Y \) are any vector fields of \( N \), then the induced metric \( g \) on \( N \) is defined by
\[
(1.11) \quad g(X, Y) = G(BX, BY).
\]
We assume that the normal bundle of \( N \) is orientable. Then we choose mutually orthogonal unit vector fields \( C_x \) normal to \( N \).

The transformations \( JBX \) and \( JC_x \) can be expressed as
\[
(1.12) \quad JBX = BfX + \eta_x(X)C_x,
\]
\[
(1.13) \quad JC_x = B\xi_x + \lambda_xC_x,
\]
A certain polynomial structure

where \( f \) is a tensor field of type (1,1), \( \eta_x \) are 1-forms, \( \xi_x \) are vector fields and \( \lambda_x \) are scalar fields defined on \( N \).

We are interested in the antinormal submanifold, that is, \( \lambda_x = 0 \) in (1.14). Then computing \( J^2 BX \), we get

\[
BX = B f^2 X + \eta_x(f X) C_x + \eta_x(X) B \xi_x,
\]

from which, comparing tangential part and normal part,

\[
f^2 X = X - \eta_x(X) \xi_x, \quad \eta_x(f X) = 0.
\]

Similarly, computing \( J^2 C_x \) we get

\[
f \xi_x = 0, \quad \eta_y(f_x) = 0.
\]

Therefore the antinormal submanifold \( N \) has a pseudo-framed structure of rank \( r \).

2. Products of pseudo-framed manifolds

Let \( M(f, \xi_x, \eta_x) \) and \( \overline{M}(\overline{f}, \overline{\xi}_a, \overline{\eta}_a) \) be two pseudo-framed manifolds of ranks \( r \) and \( \overline{r} \), respectively, where the index \( x \) runs over the range \( \{1, \ldots, n-r\} \) and the index \( a \), runs over the range \( \{1, \ldots, n-\overline{r}\} \). Now, we introduce a pseudo-framed structure on a product manifold \( M \times \overline{M} \) as follows.

For a vector field \( (X_p, \overline{X}_\overline{p}) \) of the product manifold \( M \times \overline{M} \) at a point \( (p, \overline{p}) \), we shall denote \( X_p + \overline{X}_\overline{p} \). We identify \( X \in \mathcal{T}M \) with \( X \in \mathcal{T}(M \times \overline{M}) \) by

\[(2.1) \quad \bar{X}_{(p,\overline{p})} = (X_p, 0_\overline{p}) = X_p + 0_\overline{p},\]

where \( 0_\overline{p} \) is the zero vector of \( \overline{M} \) at \( \overline{p} \). If \( \pi : M \times \overline{M} \to M \) and \( \overline{\pi} : M \times \overline{M} \to \overline{M} \) are projections \( \pi(p, \overline{p}) = p \) and \( \overline{\pi}(p, \overline{p}) = \overline{p} \), respectively, then \( \pi_* \bar{X} \in \mathcal{T}(M \times \overline{M}) \) by

\[(2.2) \quad \bar{X}_{(p,\overline{p})} = (0_p, \overline{X}_\overline{p}) = 0_p + \overline{X}_\overline{p}.
\]

Differentiable 1-forms on \( M \) and \( \overline{M} \) are identified with 1-forms on \( M \times \overline{M} \) in the same way. If \( w \) and \( \overline{w} \) are 1-forms on \( M \) and \( \overline{M} \), respectively, then a 1-form \( \bar{w} \) is defined on \( M \times \overline{M} \) by

\[(2.3) \quad \bar{w}_{(p,\overline{p})} (X_p, \overline{X}_\overline{p}) = w_p(X_p) + \overline{w}_{\overline{p}}(\overline{X}_\overline{p}).
\]

Now, for any vector fields \( X \in \mathcal{T}M_p \) and \( \overline{X} \in \mathcal{T}\overline{M}_\overline{p} \), if we put

\[(2.4) \quad F(X, \overline{X}) = (fX, f\overline{X}),
\]

then \( F \) defines a linear map of tangent space \( \mathcal{T}(M \times \overline{M}) \) onto itself. From the last equation, we get

\[(2.5) \quad F^2 = (I, \overline{I}) - (\eta_x \otimes \xi_x, 0) - (0, \overline{\eta}_a \otimes \overline{\xi}_a),
\]

where \( I \) and \( \overline{I} \) are identity tensor fields of \( M \) and \( \overline{M} \) respectively. From (2.5) we get

\[(2.6) \quad F^3 - F = 0,
\]

and \( F \) has rank \( r + \overline{r} \). If we put

\[
E_x = (\xi_x, 0), \quad E_{n-r+a} = (0, \xi_a), \quad w_x = (\eta_x, 0), \quad w_{n-r+\overline{p}} = (0, \overline{\eta}_\overline{p}),
\]

\[
E_{n-r+a} = (0, \xi_a), \quad w_{n-r+\overline{p}} = (0, \overline{\eta}_\overline{p}),
\]

\[
E_{n-r+a} = (0, \xi_a), \quad w_{n-r+\overline{p}} = (0, \overline{\eta}_\overline{p}),
\]
from which
\[ w_x(E_y) = (\eta_x(\xi_y), 0), \quad w_{n+r+a}(E_{n+r+b}) = (0, \xi_a(\eta_b)). \]
Then (2.5) can be written by
\[ F^2 = I - w_A \otimes E_A, \]
where \( I = (I, I) \) and \( A, B = 1, 2, \ldots, n + n - r - \bar{r} \).
Moreover we get
\[ FE_A = 0, \quad w_A F = 0, \quad w_A (E_B) = \delta_{AB}. \]
Thus we have

**Theorem 2.1.** Let \( M(f, \xi, \eta) \) and \( \bar{M}(\bar{f}, \bar{\xi}, \bar{\eta}) \) be pseudo-framed manifolds of ranks \( r \) and \( \bar{r} \), respectively. Then the product manifold \( M \times \bar{M} \) carries a pseudo-framed structure \((F, E_A, w_A)\) of rank \( r + \bar{r} \).

Let \( R^m \) be an \( m \)-dimensional Euclidean space. Then \( R^m \) has a trivial pseudo-framed structure \((0, \partial/dt^a, dt^a)\). Hence by Theorem 2.1 we can introduce a pseudo-framed structure on \( M \times R^m \) given by
\[ F(X, \lambda^a d/dt^a) = (fX, 0), \]
where \( \lambda^a \) are real valued functions on \( R^m \). Then we have
\[ F^2 = (I, I) - (\eta_x \otimes \xi_x, 0) - (0, dt^a \otimes d/dt^a). \]
Thus we have

**Corollary 2.2.** Let \( M(f, \xi, \eta) \) be a pseudo-framed manifold of rank \( r \) and \( R^m \) an \( m \)-dimensional Euclidean space with trivial pseudo-framed structure \((0, \partial/dt^a, dt^a)\). Then the product manifold \( M \times R^m \) has a pseudo-framed structure \((F, E_A, w_A)\) of rank \( r \) given by (2.9).

Let \( M(f, \xi, \eta) \) and \( \bar{M}(\bar{f}, \bar{\xi}, \bar{\eta}) \) be two pseudo-framed manifolds of dimensions \( n, \bar{n} \) and ranks \( r, \bar{r} \), respectively, where we assume that \( n - r = \bar{n} - \bar{r} \). For any vector fields \( X_p \in TM_p \) and \( \bar{X}_p \in T\bar{M}_p \), we define a linear map \( J \) of tangent space \( T(M \times \bar{M})_{(p, \bar{p})} \) onto itself by
\[ J(X, \bar{X}) = (fX + \eta_x(\bar{X})\xi_x, \bar{f}\bar{X} + \eta_x(X)\bar{\xi}_x). \]
Then we have
\[ J^2 = (I, I), \]
which shows that \( J \) is an almost product structure.

Thus we have

**Theorem 2.3.** Let \( M(f, \xi, \eta) \) and \( \bar{M}(\bar{f}, \bar{\xi}, \bar{\eta}) \) be two pseudo-framed manifolds. Then the product manifold \( M \times \bar{M} \) has an almost product structure \( J \) defined by (2.10).

Now, since \( R^{n-\bar{r}} \) has a trivial pseudo-framed structure \((0, \partial/dt^{\bar{a}}, dt^{\bar{a}}), \) \((t^\bar{a})\) being the coordinate in \( R^{n-\bar{r}} \), we can introduce an almost product structure \( J \) on a product manifold \( M \times R^{n-\bar{r}} \). If we put
A certain polynomial structure

(2.12) \[ J(X, \lambda \xi_x d/dt^2) = (fX + \lambda \xi_x \eta_x(X) d/dt^2), \]
then we have \( J^2 = (I, I) \).

Thus we have

**THEOREM 2.4.** Let \( M(f, \xi, \eta) \) be a pseudo-framed manifold of rank \( r \). Then the product manifold \( M \times R^{n-r} \) has an almost product structure \( J \) defined by (2.12).

Finally, we prove the following:

**THEOREM 2.5.** Let \( M(f, \xi, \eta) \) be a pseudo-framed manifold of rank \( r \). If the induced almost product structure \( J \) on \( M \times M \) is integrable, then the pseudo-framed structure \( f \) is integrable.

**Proof.** For any vector fields \( X \) and \( Y \) on \( M \times M \), we define an induced almost product structure \( J \) on \( M \times M \) as follows:

(2.13) \[ J(X, Y) = (fX + \eta_x(Y) \xi_x, fY + \eta_x(X) \xi_x). \]

Then the integrability condition of the induced almost product structure \( J \) on \( M \times M \) is given by

\[
\begin{align*}
&[J(X_1 + X_2, J(Y_1 + Y_2)] - J[J(X_1, X_2), Y_1 + Y_2] \\
&- J[J(X_1 + X_2, J(Y_1 + Y_2)] + [X_1 + X_2, Y_1 + Y_2] = 0,
\end{align*}
\]

for any vector fields \( X = X_1 + X_2 \) and \( Y = Y_1 + Y_2 \) on \( M \times M \). By a direct computation we see that the above condition is equivalent to the following:

(2.14) \[
\begin{align*}
&[f, f](X_1, Y_1) + [f X_1, \eta_x(Y_2) \xi_x] - f[X_1, \eta_x(Y_2) \xi_x] \\
&+ [\eta_x(X_2) \xi_x, f Y_1] - f[\eta_x(X_2) \xi_x, Y_1] - \eta_x([f X_2, Y_2] + [X_2, f Y_2]) \xi_x \\
&- \eta_x([\eta_x(X_1) \xi_x, Y_2] + [X_2, \eta_x(Y_1) \xi_x]) \xi_x + [\eta_x(X_2) \xi_x, \eta_x(Y_2) \xi_x] = 0,
\end{align*}
\]

(2.15) \[
\begin{align*}
&[f, f](X_2, Y_2) + [f X_2, \eta_x(Y_1) \xi_x] - f[X_2, \eta_x(Y_1) \xi_x] \\
&+ [\eta_x(X_1) \xi_x, f Y_2] - f[\eta_x(X_1) \xi_x, Y_2] - \eta_x([f X_1, Y_1] + [X_1, f Y_1]) \xi_x \\
&- \eta_x([\eta_x(X_2) \xi_x, Y_1] + [X_1, \eta_x(Y_2) \xi_x]) + [\eta_x(X_1) \xi_x, \eta_x(Y_1) \xi_x] = 0.
\end{align*}
\]

Now, putting \( X_2 = Y_2 = 0 \) in (2.14) and (2.15) we obtain

(2.16) \[ [f, f](X_1, Y_1) = 0, \]

(2.17) \[ \eta_x([f X_1, Y_1] + [X_1, f Y_1]) \xi_x = 0. \]

Again putting \( X_1 = \xi_x \) and \( Y_1 = \xi_x \) in (2.17), we get

(2.18) \[ [\xi_x, \xi_x] = 0. \]

Putting \( Y_1 = \xi_x \) in (2.16), we get

(2.19) \[ f[X_1, \xi_x] = [f X_1, \xi_x]. \]

Taking account of (2.18), (2.17) can be written by

(2.20) \[ \eta_x([f X_1, Y_1] + [X_1, f Y_1]) = 0. \]

Using (2.18), (2.19) and (2.20), the integrability conditions (2.14) and (2.15) are expressed as follows:

(2.21) \[ [f, f](X_1, Y_1) - \eta_x([\eta_x(X_1) \xi_x, Y_2] + [X_2, \eta_x(Y_1) \xi_x]) \xi_x = 0, \]
Again putting $X_1 = \xi_x$ and $Y_1 = \xi_x$ in (2.21), we get
\begin{equation}
[\eta_x, [\xi_x, Y_2] + [X_2, \xi_x]] = 0.
\end{equation}
Similarly we obtain
\begin{equation}
[\eta_x, [\xi_x, Y_1] + [X_1, \xi_x]] = 0.
\end{equation}
Then (2.21) and (2.22) are written by
\begin{align*}
[f, f](X_1, Y_1) &= 0, \quad [f, f](X_2, Y_2) = 0,
\end{align*}
which shows that the pseudo-framed structure $f$ is integrable.

3. Normal pseudo-framed structure

In the previous section, we have seen that the induced almost product structure $J$ on $M \times R^{n-r}$ is defined by
\begin{equation}
J(X, \dot{X}) = (fX + \dot{X}) \xi_x, \quad \eta_x(X) d/dt^x
\end{equation}
for any vector field $X$ on $M$ and real-valued functions $\lambda^x$ on $R^{n-r}$. We shall consider the case that the induced almost product structure $J$ is integrable.

**DEFINITION.** If the induced almost product structure $J$ on $M \times R^{n-r}$ is integrable, we say that the pseudo-framed structure $(f, \xi_x, \eta_x)$ on $M$ is normal.

Denoting by $N^A_{BC}$ the components of the Nijenhuis tensor $[J, J]$ $(X, Y)$, $N^A_{BC}$ is given by
\begin{equation}
N^A_{BC} = J^E_{A \partial E} J^C_{E \partial D} J^B_{D \partial C} - J^E_{A \partial E} J^C_{E \partial D} J^B_{D \partial C} + J^E_{A \partial E} J^D_{C \partial D} J^B_{B \partial C} - J^E_{A \partial E} J^D_{C \partial D} J^B_{B \partial C},
\end{equation}
where the indices $A, B, C, \cdots$, run over the range \{1, 2, \cdots, 2n-r\}.

Considering the Nijenhuis tensor $[J, J]$ of $J$, they computed $[J, J](X + O, Y + O)$, $[J, J](X + O, O + d/dt^x)$ and
\begin{equation}
[J, J](O + d/dt^x, O + d/dt^x),
\end{equation}
which rise to five tensors given by
\begin{align*}
N^1(X, Y) &= N^{ij}_{jk} = [f, f](X, Y) + d\eta_x(X, Y) \xi_x, \\
N^2(X, Y) &= N^{ij}_{jk} = (L_f \eta_x)(Y) - (L_f \eta_x)(X), \\
N^3(X, U) &= N^{ij}_{ja} = (L_{\xi_x} f)(X), \\
N^4(X, U) &= N^{ij}_{ja} = -(L_{\xi_x} \eta_x)(X), \\
N^5(U, V) &= N^{ij}_{xy} = L_{\xi_x} \xi_y,
\end{align*}
for any vector fields $X$ and $Y$ on $M$ and $U, V$ on $R^{n-r}$, where $L_X$ denotes the Lie derivative with respect to $X$. The pseudo-framed structure $(f, \xi_x, \eta_x)$ is normal if and only if $N^1 = 0$, that is,
\begin{equation}
N^1(X, Y) = [f, f](X, Y) + d\eta_x(X, Y) \xi_x = 0.
\end{equation}
We see that the trivial pseudo-framed structure $(O, d/dt^x, d/dt^x)$ is normal. Now, we prove the following.

**THEOREM 3.1.** Let $M$ and $\overline{M}$ be manifolds with normal pseudo-framed
structures. Then the pseudo-framed structure of the product manifold $M \times \overline{M}$ is normal.

Proof. Let $M(f, \xi_x, \eta_x)$ and $\overline{M}(\overline{f}, \overline{\xi}_\alpha, \overline{\eta}_\alpha)$ be pseudo-framed manifolds of ranks $r$ and $\bar{r}$, respectively. By Theorem 2.1 $M \times \overline{M}$ carries a pseudo-framed structure of rank $r + \bar{r}$ given by (2.4.). Then we compute

$$[F, F](X + \overline{X}, Y + \overline{Y}) = [F(X + \overline{X}), F(Y + \overline{Y})] - [F(X + \overline{X}), Y + \overline{Y}]$$

$$= [F(X + \overline{X}, F(Y + \overline{Y})] + F^2[X + \overline{X}, Y + \overline{Y}]$$

$$= [(f + \overline{f}, f + \overline{f}, f + \overline{f}, f + \overline{f} - f + \overline{f}, f + \overline{f}, f + \overline{f}), f + \overline{f}, f + \overline{f}]$$

$$= [F(X + \overline{X}, f Y + \overline{f} Y] + F^2[X + \overline{X}, Y + \overline{Y}]$$

$$= ([f X, f Y], [\overline{f} X, \overline{f} Y]) - ([f X, Y], [\overline{f} X, \overline{f} Y])$$

from which

(3.4) $[F, F] = ([f, f], [\overline{f}, \overline{f}]).$

Moreover

$$d\omega_A(X + \overline{X}, Y + \overline{Y}) E_A = \{(X + \overline{X}) \omega_A(Y + \overline{Y}) - (Y + \overline{Y}) \omega_A(X + \overline{X})$$

$$- \omega_A([X + \overline{X}, Y + \overline{Y}])\} E_A$$

$$= (X \eta_x(Y) - Y \eta_x(X) - \eta_x([X, Y]) \xi_x$$

$$+ (\overline{X} \overline{\eta}_\alpha(Y) - \overline{Y} \overline{\eta}_\alpha(X) - \overline{\eta}_\alpha([\overline{X}, \overline{Y}]) \overline{\xi}_\alpha,$$

from which

(3.5) $d\omega_A \otimes E_A = (d\eta_x \otimes \xi_x, d\overline{\eta}_\alpha \otimes \overline{\xi}_\alpha).$

From (3.4) and (3.5) we get

(3.6) $N^1(F) = (N^1(f), N^1(\overline{f})),$

which shows that $M \times \overline{M}$ has a normal pseudo-framed structure.

Lemma 3.2. If a pseudo-framed structure $(f, \xi_x, \eta_x)$ is normal on $M$, then we have

1. $d\eta_x(X, \xi_y) = 0,$
2. $[\xi_x, \xi_y] = 0,$
3. $f[X, \xi_x] = [f X, \xi_x],$
4. $d\eta_x(f X, Y) - d\eta_x(X, f Y) = 0.$

Proof. Putting $Y = \xi_y$ in (3.3), we get

(3.7) $- f[f X, \xi_y] + f^2[X, \xi_y] + d\eta_x(X, \xi_y) \xi_x = 0.$

Taking the inner product of the left hand side of the equation by $\xi_x$, we obtain

(3.8) $d\eta_x(X, \xi_y) = 0.$

Secondly, Putting $X = \xi_x$ and $Y = \xi_y$ in (3.3), and using (3.8) we get

(3.9) $[\xi_x, \xi_y] = 0.$

Thirdly, from (3.7) and (3.8) we get
\[(3.10)\quad f[X, \xi] = f^q[fX, \xi] - [fX, \xi] - \eta_x([fX, \xi]) \xi_x = [fX, \xi],\]

with the help of (3.8). Fourthly, Putting \(Y = fY\) in (3.3), we get
\[
[fX, f^2Y] - f[fX, fY] - [fX, f^2Y] + f^2[X, f Y] + d\eta_x(X, f Z) \xi_y = 0,
\]
from which, taking the inner product of the last equation by \(\xi_x\)
\[
\eta_x([fX, f^2Y]) + d\eta_x(X, f Y) = 0,
\]
or
\[(3.11)\quad \eta_x([fX, Y]) - fX(\eta_x(Y)) + d\eta_x(X, f Y) = 0.
\]
On the other hand, by the definition of \(d\eta_x\) we get
\[(3.12)\quad fX(\eta_x(Y)) - Y(\eta_x(fX)) - \eta_x([fX, Y]) - d\eta_x(fX, Y) = 0.
\]
Adding the last two equations we have
\[(3.13)\quad d\eta_x(X, f Y) - d\eta_x(fX, Y) = 0.
\]
By the definition of Lie derivative, (1), (2), (3) and (4) are equivalent to \(N^2 = 0\), \(N^5 = 0\), \(N^3 = 0\) and \(N^2 = 0\), respectively.
Thus we have also the following (cf. [11]): If a pseudo-framed structure is normal, that is, \(N^1 = 0\), then we have
\[
N^2 = N^3 = N^4 = N^5 = 0.
\]

Finally, we prove the following.

**Theorem 3.3.** Let \(M(f, \xi, \eta)\) be a manifold with normal pseudo-framed structure of rank \(r\). If \(f\) and \(\eta\) are Killing tensors, the structure tensors \(f\), \(\xi\) and \(\eta\) are covariantly constant, that is,
\[
\nabla_X \xi = 0, \quad \nabla_X \eta = 0.
\]

**Proof.** Since \(\eta\) are Killing forms we get
\[
(\nabla_X \eta) (Y) + (\nabla_Y \eta) (X) = 0,
\]
from which
\[(3.14)\quad d\eta_x(X, Y) = -2(\nabla_Y \eta)(X) - 2g(X, \eta_x(Y)).
\]
By the normality \(N^3\) vanishes identically, that is, \(L_\xi f = 0\), and hence we get
\[
(L_\xi F)(X, Y) = (L_\xi g)(X, f Y) = 0,
\]
from which
\[
(\nabla_\xi F)(X, Y) = (\nabla_X F)(Y, \xi) + (\nabla_Y F)(X, \xi).
\]
Since \(F\) is a Killing tensor, we get
\[(3.15)\quad (\nabla_\xi F)(X, Y) = 0.
\]
Since \(f\) is a Killing tensor, by the normality \(N^3 = 0\), we get
\[
0 = (\nabla_\xi f)(X) - (\nabla_X f)(\xi) = (\nabla_X \xi)(f) = (\nabla_X \xi)(f).
\]
Hence if \(X\) is orthogonal to \(\xi_x\), then we can put \(X = fZ\) for some \(Z\) and we obtain
\[
d\eta_x(X, Y) = -2g(X, \eta_x(Y)) = -2g(fZ, \eta_x(Y)) = -2g(Z, f(\nabla_Y \xi))(Y, \eta_x(Y)) = 0.
\]
Thus, from (3.8) we have
\[(3.16)\quad d\eta_x = 0.
\]
A certain polynomial structure

From (3.14) we get

\[ \mathcal{V}_x \eta_x = 0, \]

from which

\[ \mathcal{V}_x \xi_x = 0. \]

On the other hand, by the normality and (3.16) we get

\[ (\mathcal{V}_f X f) Y - (\mathcal{V}_x f) X - f(\mathcal{V}_x f) X + f(\mathcal{V}_y f) X = 0. \]

Since \( f \) is a Killing tensor, we get

\[ - (\mathcal{V}_Y f) X + (\mathcal{V}_y X f) Y - 2 f(\mathcal{V}_x f) Y \]

\[ = - (\mathcal{V}_f X f) + f(\mathcal{V}_x f) X + (\mathcal{V}_x f) Y - f(\mathcal{V}_x f) Y - 2 f(\mathcal{V}_x f) Y = 0, \]

from which \( f(\mathcal{V}_x f) Y = 0. \)

Applying \( f \) to the last equation, we get

\[ (\mathcal{V}_x f) Y - \eta_x((\mathcal{V}_x f) Y) \xi_x = 0, \]

from which we have

\[ \mathcal{V}_x f = 0. \]

References

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