DETERMINANTS OF n-DIMENSIONAL MATRICES

BY JIN B. KIM AND JAMES E. DOWDY

1. Introduction

We denote by $M_n(p)$ the set of all $p \times p \times \cdots \times p = p^n$ matrices over the real numbers. Any matrix $A$ in $M_n$ is called an $n$-dimensional matrix. If $A$ is a square matrix or an $p \times p$ matrix, then $A$ is a 2-dimensional matrix. There are many kinds of determinants (see [1] and [2] for definitions of determinants) for $n$-dimensional matrices (where $n$ is a positive integer greater than 2).

We first define a determinant $\det_j(A)$ of an $n$-dimensional matrix $A$ (for $j = 1, 2, \ldots, n$) following the (Japanese language) paper [1] by R. Kaneiwa. We also define a product $AB$ of two $2m$-dimensional matrices $A$ and $B$ in $M_{2m}(p)$. We prove that the associative law holds for the product $AB$ in $M_{2m}(p)$. We obtain the identity matrix of $M_{2m}(p)$. We study elementary properties of $\det_j(A)$ for $A \in M_n(p)$, prove that $\det_j(AB) = \det_j(A) \det_j(B)$ and compute determinants of identity matrices as $\det_j(I) = (p!)^{m-1}$, where $I$ is the identity matrix in $M_{2m}(p)$.

2. Definition of a determinant

In this section we give a definition of a determinant $\det_j(A)$ of an $n$-dimensional matrix $A$ in $M_n(p)$, for $j = 1, 2, \ldots, n$. We assume all matrices are real matrices. Let $p$ be a positive integer greater than 1. $S(p)$ denotes the symmetric group on the set $\{1, 2, \ldots, n\}$. A matrix $A$ is called an $n$-dimensional (square) matrix over $S(p)$ if $A$ is an $p \times p \times \cdots \times p = p^n$ matrix. We denote by $M_n(p)$ the set of all $n$-dimensional matrices over $S(p)$. (A 2-dimensional square matrix $A$ over $S(p)$ is a $p \times p$ matrix and we call it just a square matrix). We define a set $S^*(p) = \{\lambda = (\lambda_1 \lambda_2 \cdots \lambda_n) : \lambda_i \in S(p)\}$. We define $\pi \lambda$ for $\pi \in S(p)$ and $\lambda = (\lambda_1 \lambda_2 \cdots \lambda_n) \in S^*(p)$ by $\pi \lambda = (\pi \lambda_1 \pi \lambda_2 \cdots \pi \lambda_n)$. We define a relation $R$ on $S^*(p)$ by $\lambda R \mu$ iff $\mu = \pi \lambda$ (for some $\pi \in S(p)$), for $\lambda, \mu \in S^*(p)$. It is clear that $R$ is an equivalence relation on $S^*(p)$ and we define $S^*(p)/R = S_n(p)$ as the set of all equivalence classes defined by $R$. Let $A \in M_n(p)$. An entry of $A$ takes the form $a_{i_1 i_2 \cdots i_n}$. Let

Received June 5, 1980
\( \pi = (1, 2, \ldots, p) \in S(p) \). Then we can write \( \pi(i) = i' \). Letting \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_n(p) \), define

\[
\alpha_1 = \prod_{i=1}^{p} \alpha_{i(i)} = \alpha_{1(1)} \alpha_{2(1)} \cdots \alpha_{n(1)} \alpha_{1(2)} \alpha_{2(2)} \cdots \alpha_{n(2)} \cdots \alpha_{1(p)} \alpha_{2(p)} \cdots \alpha_{n(p)},
\]

as a product of \( p \) entries of the matrix \( A \).

**Lemma 1.** Let \( \lambda \in S_n(p) \) and let \( A \in M_n(p) \). Then \( \alpha_1 = a_\lambda \), for \( \mu = [\lambda] \).

**Remark** \( a_\lambda = a_\mu \) means that \( a_\lambda \) and \( a_\mu \) are identical when we apply the commutativity of the real numbers.

**Proof.** We prove it by induction on \( n \). If \( n = 1 \), it is trivial. Assume that we have proved it for \( n < k \), where \( k \) is a fixed positive integer greater than 1. Let \( n = k \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) = (\pi \lambda_1, \pi \lambda_2, \ldots, \pi \lambda_k) \) for \( \mu = [\lambda] \). Without loss of generality we can assume that \( \pi = (1, i) \) (a transposition and \( i \neq 1 \)) and \( \lambda_1 = I \), the identity of the group \( S(p) \). We consider \( (\pi \lambda_1, \pi \lambda_2, \ldots, \pi \lambda_k) = I = (1) \). We see that \( \pi \lambda_1(1) = i \) and by inductional assumption we have that \( I_1 = (\pi \lambda_2(t), \pi \lambda_3(t), \ldots, \pi \lambda_k(t)) \) for some \( t \). We have \( \pi \lambda_j(1) = \lambda_j(t) \) \((j = 2, 3, \ldots, k)\) from which we get that \( \lambda_j(t) = \lambda_j(t) \) and hence we have \( a_{\lambda_1(1)} = a_{\lambda_2(t)} \cdots a_{\lambda_k(t)} = a_{\mu(1)} \). Now consider \( I_2 = (\pi \lambda_1(i), \pi \lambda_2(i), \ldots, \pi \lambda_k(i)) = (1, \pi \lambda_2(i), \pi \lambda_3(i), \ldots, \pi \lambda_k(i)). \) By inductional assumption we can have that \( I_2 = (\pi \lambda_1(t), \pi \lambda_2(t), \pi \lambda_3(t), \ldots, \pi \lambda_k(t)) \) from which we obtain that \( \lambda_j(i) = \lambda_j(t) \) and \( \pi \lambda_j(i) = \lambda_j(t) \) and hence we have that \( I_2 = (\lambda_1(1), \lambda_2(1), \ldots, \lambda_k(1)) \). Therefore we have \( a_{\lambda_1(1)} = a_{\mu(i)} \). Finally we consider \( K_3 = (\pi \lambda_1(j), \pi \lambda_2(j), \ldots, \pi \lambda_k(j)) \) for \( 1 \neq j \neq i \). We can show that \( K_3 = (\lambda_1(j), \lambda_2(j), \ldots, \lambda_k(j)) \) and hence \( a_{\lambda_1(1)} = a_{\mu(j)} \) and \( a_{\lambda_1(i)} = a_{\mu(i)} \). This proves the lemma.

By Lemma 1, we define \( a_{\lambda(j)} \) as \( a_{\lambda(j)} = a_{\mu(j)} \) for \( \mu = [\lambda] \). Now we define \( \text{sign}_j([\lambda]) \) for \( [\lambda] \in S_n(p) \). For \( j \) there exists \( u = (u_1, u_2, \ldots, u_n) \in [\lambda] \) such that \( u_j = I \), the identity of the group \( S(p) \). \( \text{sign}_j([\lambda]) = \prod_{i=1}^{n} \text{sign}(u_i) \) is defined as the product of all \( \text{sign}(u_i) \).

We have now a definition of a determinant of \( A \).

**Definition 1.** Let \( A = (a_{ij}) \in M_n(p) \) be an \( n \)-dimensional matrix over \( S(p) \) and \( A \) is a real matrix. We define

\[
\text{det}_j(A) = \sum_{[\lambda]} \text{sign}([\lambda]) a_{\lambda(j)}
\]

the summation being taken for all elements \([\lambda]\) in \( S_n(p) \). We may call \( \text{det}_j(A) \) a \( j \)-determinant of an \( n \)-dimensional matrix.

Note that if \( A \) is a \( p \times p \) matrix, then \( \text{det}_1(A) = \text{det}_2(A) = \text{det}(A) \). \(|S|\) denotes the cardinality of a set \( S \).
Lemma 2. Let $A$ be as in Definition 1. Then $\det_j(A)$ has $(p!)^{n-1}$ terms in its expansion, $j=1, 2, \ldots, n$.

Proof. We can see that $|S_n(p)| = (p!)^{n-1}$.

3. Elementary properties of determinants

In this section we shall prove that if $n$ is even, then $\det_j(A) = \det_1(A)$ for all $j=2, 3, \ldots, n$. In the case $n=2m$, we just write $\det(A)$ instead of $\det_j(A)$.

Example 1. Let $A$ be a 3-dimensional matrix over $S(3)$ and let

$$A = \begin{bmatrix}
a_{111} & a_{112} & a_{113} & a_{211} & a_{212} & a_{213} & a_{311} & a_{312} & a_{313} \\
a_{121} & a_{122} & a_{123} & a_{221} & a_{222} & a_{223} & a_{321} & a_{322} & a_{323} \\
a_{131} & a_{132} & a_{133} & a_{231} & a_{232} & a_{233} & a_{331} & a_{332} & a_{333}
\end{bmatrix}$$

with $a_{112}=1$, $a_{123}=4$, $a_{122}=7$, $a_{211}=6$, $a_{212}=5$, $a_{221}$ and $a_{333}=3$, and all other entries of $A$ are zero. Let $[\lambda] = [(I (I (12)))]$, $[u] = [(I (I (12) I))]$ and $[v] = [(I (12) I I)]$. Then we see that

$$\text{sign}_1[\lambda] = \text{sign}_2[\lambda] = \text{sign}_1[u] = \text{sign}_3[u] = \text{sign}_2[v] = \text{sign}_3[v] = -$$

$$\text{sign}_3[\lambda] = \text{sign}_2[u] = \text{sign}_1[v] = +, \quad a_{[1]} = 6, \quad a_{[u]} = 60 \quad \text{and} \quad a_{[v]} = 42.$$  

We can see that $a_{[\lambda]}$, $a_{[u]}$ and $a_{[v]}$ are only non-zero terms of the expansion of each $\det_jA$, $\det_1A = -24$, $\det_2A = 12$ and $\det_3A = -96$.

Theorem 1. Let $n=2m$ $(m \geq 2)$ and let $A=(a_{ij} \ldots k) \in M_n(p)$. Then $\det_1(A) = \det_j(A)$ for $j=2, 3, \ldots, n$.

Proof. Let $[\lambda]$ be an arbitrary member of $S_n(p)$ and consider $\text{sign}_1([\lambda])$. Without loss of generality we can assume that $\text{sign}_1([\lambda]) = +$, $\lambda = (\lambda_1 \lambda_2 \ldots \lambda_n)$ and $\lambda_1 = 1$. We suppose that $\text{sign}_j([\lambda]) = - (j \neq 1)$, $\pi \lambda = u = (u_1 u_2 \ldots u_n) = (\pi \lambda_1 \pi \lambda_2 \ldots \pi \lambda_n)$ and $u_j = 1$. Note that $\text{sign}_1([\lambda]) = \text{sign}(\lambda) = \prod_{i=1}^n (\text{sign} \lambda_i)$ and $\text{sign}_j([\lambda]) = \text{sign}(u) = -$. From $\text{sign}(u) = -$, there are $u_{i(1)}$, $u_{i(2)}$, $u_{i(2q+1)}$ such that $\text{sign}(u_{i(1)}) = -$ and $\text{sign}(u_{i(2q+1)}) = +$. For $s \neq i$ $(i = 1, 2, \ldots, 2q+1)$.

We can see that $+ = \text{sign}(\lambda) = \prod_{i=1}^n (\text{sign} \lambda_i) = \prod_{i=1}^n (\text{sign}(\pi^{-1}u_{i})) = \prod_{i=1}^n (\text{sign}(u_{i})) = \text{sign}(u) = -$, a contradiction. This proves the theorem.

Theorem 2. Let $A=(a_{ij} \ldots k) \in M_{2m+1}(p)$. Then there are $2m+1$ distinct determinants $\det_j(A)$, $j=1, 2, \ldots, 2m+1$.

Proof. For $n=3$, see Example 1. Let $n=2m+1$ $(m \geq 2)$. We need notations. Let $V_n(p) = \{(i_1 i_2 \ldots i_n) : i_j$ is a positive integer such that $1 \leq i_j \leq p\}$ and $d_{ij}$ denotes the Kronecker's delta $(d_{ij} = 1$ if $i=j$ and $d_{ij} = 0$ if $i \neq j)$. Let $d_i = (d_{i1}d_{i2} \ldots d_{in}) \in V_n(p)$. In $V_n(p)$, define $e(1) = d_1 + d_2 + \cdots + d_n = (11 \ldots 1)$,
$e(i) = ie(1) = (i \ldots i)$, $e(1 \cdot i) = e(1) + d_1$ and $e(2 \cdot i) = e(2) + d_2$. With these vectors we define all non-zero entries of $A$ as follows. $a_{e(i)} = 1$ for all $i \geq 3$, $a_{e(1 \cdot n)} = a_{e(2 \cdot n)} = 1$, $a_{e(1 \cdot n-1)} = a_{e(2 \cdot n-1)} = \sqrt{2}$, $\ldots$, $a_{e(1 \cdot n-i+1)} = a_{e(2 \cdot n-i+1)} = \sqrt{i}$, $\ldots$, $a_{e(1 \cdot 1)} = a_{e(2 \cdot 1)} = \sqrt{n}$ (and all other entries of $A$ are equal to 0). We now define $\lambda(i) = (\lambda_1 \lambda_2 \ldots \lambda_n) \in S^n(p)$ as follows: $\lambda_{n-i+1} = (1 \cdot 2)$, a transposition, and $\lambda_i = I$ (the identity) for $t \neq n - i + 1$. Then we can see that

$$\text{sign}_j([\lambda(i)]) = \begin{cases} + & \text{if } i = n - j + 1, \\ - & \text{otherwise.} \end{cases}$$

With these data we can compute $\det_j(A)$ and obtain that $\det_j(A) = -(n+1)n/2 + 2(n-j+1)$. We can check that all $\det_j(A)$ are distinct for $j = 1, 2, \ldots, n=2m+1$. This proves the theorem.

We shall establish a theorem which is analogous to that if any two rows of a matrix $A$ are identical then $\det A = 0$ for a 2-dimensional matrix $A$. To do this we introduce notations. We recall that $V_n(p) = \{ (i_1 i_2 \ldots i_n) : i_j \text{ are integers such that } 1 \leq i_j \leq p \}$ Letting $\lambda \in V_n(p)$, $a_{\lambda}$ denotes an entry of $A = (a_{ij} \ldots k) \in M_n(p)$. (Note that we have used $a_{\lambda} (\lambda \in S^n(p))$ as a product of $p$ entries of $A$ in the section 2). We define $\lambda(i,j) = (\lambda_1 \lambda_2 \ldots \lambda_n) \in V_n(p)$ by $\lambda_i = j$. Let $A = (a_{ij} \ldots k) \in M_n(p)$. Define $A_{m+1} = (a_{(m+1)})$ a submatrix of $A$, and we call $A_{m+1}$ the $i$th row (or face) of $A$ in the $m$-direction. For simplicity, we often denote $A_{1\cdot i}$ by $A_{i}$ and we may call $A_{i}$ the $i$th row (or face) of $A$.

**Theorem 3.** Let $A = (a_{ij} \ldots k) \in M_n(p)$.

(1) Let $B = (b_{ij} \ldots k) \in M_n(p)$ be the matrix obtained from $A$ by multiplying row $i_0$ of $A$ by scalar $r$ (that is, $B_i = A_{i_0}$, $i \neq i_0$ and $B_{i_0} = rA_{i_0}$). Then $\det_j B = r\det_j A$.

(2) Let $B$ be obtained from $A$ by interchanging the $i$th row and the $k$th row of $A$ (that is, $B_i = A_k$, $B_k = A_i$ and $B_{t} = A_{t}$ for $B = (B_1 B_2 \ldots B_p)$. Then $\det_j B = -\det_j A$.

**Proof.** We omit the proof of (1) and we consider (2). Let $[\lambda]$ be an arbitrary member of $S_n(p)$ and without loss of generality we can assume that $\lambda_1 = 1$ for $\lambda = (\lambda_1 \lambda_2 \ldots \lambda_n)$. Define $u = (u_1 u_2 \ldots u_n)$ by $u_1 = (i \cdot k)$ and $u_t = \lambda_t$ ($t \neq 1$). Then $\text{sign}_1([\lambda]) = -\text{sign}_1([u])$ and we can show that

$$\det_j B = \sum_{[u] \in S_n(p)} \text{sign}_1([u]) b_{[u]} = -\sum_{[\lambda] \in S_n(p)} \text{sign}_1([\lambda]) a_{[\lambda]} = -\det_j A$$

This proves the theorem.

**4. A Product of two matrices**

For $A = (a_{ij} \ldots k)$, $B = (b_{ij} \ldots k) \in M_{2m}(p)$, we define a product $AB = C =$
Determinants of \( n \)-dimensional matrices

\((c_{ij...k})\) of two matrices \( A \) and \( B \) as follows:
\[
c_{i_1i_2...i_{2m}} = \sum_{i_1' \leq i_2' \leq ... \leq i_{2m}} (a_{i_1'i_2'...i_{2m}}) (b_{i_1'i_2'...i_{2m}})
\]

We can see that \( C \in M_{2m}(p) \).

**Lemma 3.** Let \( M_{2m}(p) \) be the set of all \( 2m \)-dimensional matrices over \( S(p) \). Then \( (AB)C = A(BC) \) for all \( A, B, C \) in \( M_{2m}(p) \). Thus \( M_{2m}(p) \) forms a semigroup under the matrix product defined in the above.

We omit the proof of the lemma. We define a matrix.

**Definition 2.** Let \( \lambda \in V_m(p) \). For \( \lambda \) we use a notation \( \lambda \lambda = (\lambda \lambda) \in V_{2m}(p) \) as a vector with \( 2m \) components. Let \( B = (b_{ij...k}) \) be a matrix in \( M_{2m}(p) \) defined as follows: \( b_{\lambda \lambda} = 1 \) for \( \lambda \in V_m(p) \) and all other \( b_{uv} = 0 \) (\( u \neq \lambda \lambda \)). We denote this \( B \) by \( I \) and we call it the identity matrix of the semigroup \( M_{2m}(p) \).

**Lemma 4.** Let \( I \) be a matrix defined in Definition 2. Then \( IA = AI = A \) for all \( A \in M_{2m}(p) \).

**Proof.** Let \( I = (b_{ij...k}), A = (a_{ij...k}) \) and \( AI = C = (c_{ij...k}) \). Then
\[
c_{i_1i_2...i_{2m}} = \sum_{i_1' \leq i_2' \leq ... \leq i_{2m}} (a_{i_1'i_2'...i_{2m}}) (b_{i_1'i_2'...i_{2m}})
\]
\[
= a_{i_1'i_2'...i_{2m}} b_{i_1'i_2'...i_{2m}}
\]
\[
= a_{i_1'i_2'...i_{2m}} b_{i_1'i_2'...i_{2m}}
\]

since \( b_{\lambda \lambda} = 1 \) and \( b_{uv} = 0 \) (\( u \neq v \)). We can prove that \( IA = A \). This proves the lemma.

Combining Lemmas 3 and 4 we have the following.

**Theorem 4.** \( M_{2m}(p) \) is a semigroup with the identity \( I \).

**Theorem 5.** Let \( A, B \in M_{2m}(p) \). Then \( \det(AB) = \det A \det B \).

**Proof.** Let \( I \) be the identity of \( M_4(3) \). Then we can compute that \( \det I = 6 \). This proves the theorem.

## 5. Determinants of identity matrices

We shall prove the following theorem.

**Theorem 6.** Let \( I \) be the identity matrix of the semigroup \( M_{2m}(p) \). Then \( \det(I) = (p!)^{m-1} \).

**Remark.** In the proof of Theorem 5, we mentioned that, for \( I \in M_4(3) \), \( \det I = 3 = 6 \), which is a part of Theorem 6. For the identity matrix \( I \) in \( M_2(p) \), we know that \( \det I = 1 \), which is also a part of Theorem 6.
Proof. Define $V_1=\{(\lambda_1, \lambda_2, \ldots, \lambda_m) : \lambda_1=1\}$. Similarly, we define $V_i=\{(\lambda_1, \lambda_2, \ldots, \lambda_m) : \lambda_i=i\}$. Let $I=(a_{ij})$ be the identity matrix of the semigroup $M_{2m}(p)$. Then any non-zero entry of $I$ is of the form $a_{ij}$ $(\lambda \in V_m(p))$. Define $E(I) = \{a_{ij} : \lambda \in V_m(p)\}$ as the set of all non-zero entries of $I$. Note that $a_{ii}=1$. Define $I_1=\{a_{ij} \in E(I) : \lambda \in V_1\}$. Then we can see that $|I_1|=p^{m-1}$. We recall that $e(1)=(11 \ldots 1) \in V_m(p)$. Let $B=\bar{a}_1 \bar{a}_2 \ldots \bar{a}_p$ be a term of the expansion of the determinant of $I$. We can pick $\bar{a}_1$ from $I_1$ and we can assume that $\bar{a}_1=ae(1)e(1)$. For $ae(1)e(1)$, we define $U_2=\{\lambda=(2 \lambda_2 \lambda_3 \ldots \lambda_m) \in V_2 : \lambda_i \geq 2\}$ and define $I_2=\{a_{ij} \in E(I) : \lambda \in U_2\}$. We can see that $|I_2|=(p-1)^{m-1}$. We can see that $\bar{a}_2$ must be a member of $I_2$. We can assume (without loss of generality) that $\bar{a}_2=ae(2)e(2)$, where $e(2)=2e(1)=(22 \ldots 2) \in V_m(p)$. For $B=ae(1)e(1)ae(2)e(2)\bar{a}_3 \ldots \bar{a}_p$, we define $U_3=\{\lambda=(\lambda_1 \lambda_2 \ldots \lambda_m) \in V_3 : \lambda_i \geq 3, \ i \neq 1\}$ and define $I_3=\{a_{ij} : \lambda \in U_3\}$. Note that $|I_3|=(p-2)^{m-1}$. We see that $\bar{a}_3$ belongs to $I_3$. Inductively, for $\bar{a}_i=ae(i)e(i)$, we define $U_{i+1}=\{\lambda=(\lambda_1 \lambda_2 \ldots \lambda_m) \in V_{i+1} : \lambda_i \geq i+1\}$ and define $I_{i+1}=\{a_{ij} \in E(I) : \lambda \in U_{i+1}\}$. Then we can show that $|I_{i+1}|=(p-i)^{m-1}$ and $\bar{a}_{i+1}$ must be a member of $I_{i+1}$. Therefore we can say that the total number of such terms $B=\bar{a}_1 \bar{a}_2 \ldots \bar{a}_p$ in the expansion of the determinant of $I$ is equal to $(p!)^{m-1}$ because of that every term $B$ takes the $+$ sign, that is, $B=1$. This proves the theorem.

Problem. Prove or disprove that \(\det(AB)=c(\det(A))(\det(B))\), where $c$ is a constant and $A, B \in M_{2m}(p)$.

References


West Virginia University