COMPACTNESS OF HOMOGENEOUS SPACES WITH FINITE VOLUMES

BY JAIHAN YOON AND KWANG SIK JEONG

Let \( G \) be a locally compact group and \( H \) be a closed subgroup such that \( G/H \) admits a finite \( G \)-invariant measure. Then under suitable restrictions on \( G \) or \( H \) it ensures \( G/H \) to be compact. For example, such is the case when \( G \) is a connected Lie group and \( H \) is any closed subgroup with finitely many connected components [4]. Also K.C. Sit generalized the above Mostow's result and proved [5] that \( G/H \) is compact whenever \( G \) is a locally compact and \( \sigma \)-compact group with the open identity component and \( H \) is the fixed points of a set of automorphisms and \( G/H \) admits a finite invariant measure.

In this paper we prove the following:

**Theorem.** Let \( G \) be a \([C]\)-group and \( H \) the centralizer of an element of \( x \) of \( G \) such that \( D = \{ gxg^{-1} : g \in G \} \) is closed. If \( G/H \) admits a finite invariant measure, then \( G/H \) is compact.

A simple example provides a \([C]\)-group \( G \) with non-open identity component; \( G = H \times K \) where \( H \) is any connected locally compact group and \( K \) is a compact, non-discrete and totally disconnected locally compact group.

§ 1. Preliminary lemmas.

For a locally compact group \( G \), \( G_0 \) will denote the identity component of \( G \) and the group \( G \) will be called a \([C]\)-group if the quotient group \( G/G_0 \) is compact. It is well known that a \([C]\)-group \( G \) can be approximated by a Lie groups; each neighbourhood of the identity contains a compact normal subgroup \( K \) of \( G \) such that \( G/K \) is a Lie group.

We shall modify this well known approximation theorem so that we can apply directly in proving our theorem.

We know that each neighbourhood of the identity of a compactly generated locally compact group \( G \) contains a compact normal subgroup \( H \) such that the quotient group \( G/H \) satisfies the second axiom of countability [1, p 71]. In particular, this is true for \([C]\)-groups. Now let \( K \) (resp. \( H \)) be a compact normal subgroup of a \([C]\)-group such that \( G/K \) is a Lie group (re-
sp. $G/H$ satisfies the second axiom of countability). Then $HK$ is a compact normal subgroup and the second countable group $G/HK$ is isomorphic (topologically) to $(G/K)/(HK/K)$ which is a Lie group. Thus we have

**Lemma 1.** A $[C]$-group can be approximated by second countable Lie group.

A locally compact space $X$ is called a homogeneous $G$-space if $G$ acts on $X$ transitively. Thus $G/H$ is a homogeneous $G$-space for any closed subgroup $H$ by a left translation. A regular Borel measure $\mu$ on $X$ is $G$-invariant if $\mu(gE) = \mu(E)$ for each Borel measurable set $E$ and $g \in G$.

The following lemma is proved by Greenleaf, Moskowitz and Rothschild [2, p.151].

**Lemma 2.** Let $G$ be a second countable Lie group and $A$ be the fixed points of a set of automorphisms of $G$. If $G/A$ admits a $G$-invariant measure, then $G/A$ is compact.

From now on a locally compact group will be assumed to be $\sigma$-compact unless otherwise specified. Let $G$ (resp. $G'$) be a locally compact group and let $X$ (resp. $X'$) be a homogeneous $G$-space (resp. $G'$-space).

**Lemma 3.** If $\pi : G \to G'$ is an open and continuous epimorphism and $\eta : X \to X'$ an equivariant continuous surjection, then $\eta$ is an open mapping and a finite $G$-invariant measure $\mu$ on $X$ can be transformed into a finite $G'$-invariant measure $\mu'$ on $X'$.

Although this lemma is well known, we sketch the proof for the convenience sake.

The proof of the openness of $\eta$ is based on the fact that a continuous surjection from a locally compact and $\sigma$-compact group to a Baire space (which is also a $G$-space) is open [3, p.39]. Applying this fact to the mapping

$$f : G' \to X' ; \quad g \to g \cdot \eta(x),$$

we see that $f$ is open. Now the equality, for any neighbourhood $V$ of the identity of $G$, $\pi(V)\eta(x) = \eta(Vx)$ proves that $\eta$ is open [5]. We show that $\mu'$, defined on $X'$ by $\mu'(E) = \mu(\eta^{-1}(E))$ for every Borel set, is a regular measure. The $\mu'$ is clearly a measure. Since $\mu'$ is finite on $X$, it suffices to show that

$$\mu'(E) = \sup \{ \mu'(K) : K' \text{ is compact, } K' \subset E \}$$

for each measurable set $E$ in $X'$. Clearly we have

$$\mu'(E) \geq \sup \{ \mu'(K') : K' \text{ is compact, } K' \subset E \} \geq$$
Compactness of homogeneous spaces with finite volumes  

\[ \sup \{ \mu(\eta^{-1}(K')) : K' \text{ is compact}, \eta^{-1}(K') \subset \eta^{-1}(E) \} \geq \sup \{ \mu(\eta^{-1}(K')) : \eta^{-1}(K') \subset \eta^{-1}(E), K' \text{ is compact} \} \]

For a compact set \( K \subset \eta^{-1}(E) \), \( \eta(K) \) is compact and \( \mu(K) \leq \mu(\eta^{-1}(\eta(K))) \) and it follows that

\[ \sup \{ \mu(\eta^{-1}(K')) : \eta^{-1}(K') \subset \eta^{-1}(E), K' \text{ is compact} \} \geq \sup \{ \mu(K) : K \text{ is compact}, K \subset \eta^{-1}(E) \}. \]

The second term of the inequality is, by the regularity of \( \mu \), \( \mu(\eta^{-1}(E)) \) and this is \( \mu'(E) \) by the definition of \( \mu' \). Thus we have shown that \( \mu'(E) = \sup \{ \mu'(K') : K' \text{ is compact}, K' \subset E \} \), the regularity of \( \mu' \). The \( G' \)-invariance of \( \mu \) follows from the fact that \( \eta \) is an equivariant.

**Lemma 4.** [4, Lemma 2.5] Let \( H \subset F \) be closed subgroups such that \( G/H \) admits a finite \( G \)-invariant measure \( \mu \). Then \( G/F \) and \( F/H \) admits, respectively, finite \( G \)-invariant and \( F \)-invariant measures of which \( \mu \) is a product.

§ 2. The Proof of Theorem

By Lemma 1, there is a compact normal subgroup \( K \) such that \( G/K \) is a second countable Lie group. Since \( KH \supseteq K \) and \( KH \) is closed, \( G/KH \) admits a finite invariant measure (Lemm 4). Since \( (G/H)/(KH/H) \) is homeomorphic to \( G/KH \) and \( KH/H \) is compact, \( G/H \) is compact if and only if \( G/KH \) is compact. Thus we reduced the problem to “whether \( G/KH \) is compact provided \( G/KH \) admits a finite \( G \)-invariant measure”.

Let \( \lambda \) and \( \lambda' \) be the usual actions of \( G \) on \( G/KH \) and \( G/K \) on \( (G/K)/(KH/K) \), respectively. Then as the diagram shown below, there corresponds a continuous surjection (in fact, a homeomorphism) \( \eta : G/KH \rightarrow (G/K)/(KH/K) \) defined by \( \eta : xKH \rightarrow xK(HK/K) \) so that the diagram commutes, i.e., \( \eta \) is an equivariant mapping.

\[
\begin{array}{ccc}
G \times G/KH & \longrightarrow & G/HK \\
\downarrow \pi \times \eta & & \downarrow \eta \\
G/K \times ((G/K)/(KH/K)) & \longrightarrow & (G/K)/(KH/K)
\end{array}
\]

In the diagram, \( \pi \) denotes the canonical projection of \( G \) onto \( G/K \). Therefore, by Lemma 2 the \( G \)-invariant finite measure on \( G/KH \) induces a finite \( G/K \)-invariant measure on \( (G/K)/(KH/K) \).

Let \( H' \) be the centralizer of \( xK \) in \( G/K \). Since \( \pi^{-1}(H') = \{ g \in G : g^{-1}x^{-1}gx \in K \} \) and contains \( H \), \( HK/K \subset H' \) and \( H'/HK/K \) admit finite invariant measures.
Since $G/K$ is second countable Lie group, we can apply Lemma 2 and deduce that $(G/K)/H$ is compact.

Note that $(G/K)/H'$ is homeomorphic to $((G/K)/(KH/K))/((H'/(KH/K))$. Therefore, the compactness of $G/HK$ (which is homeomorphic to $(G/H)/(KH/K)$) follows from the compactness of $H'/(KH/K)$ which remains to be shown.

Since $H'/(KH/K) \cong \pi^{-1}(H')/KH$ is a continuous image of $\pi^{-1}(H')/H$, it suffices to show that $\pi^{-1}(H')/H$ is compact. Consider a continuous map $j_x$ on $G$ defined by $j_x(g) = g(x)g^{-1}$, $g \in G$. Then $j_x^{-1}(xK) = \pi^{-1}(H')$ and $j_x^{-1}(x) = H$. Therefore the restriction $f$ of $j_x$ to $\pi^{-1}(H')$ is continuous on $\pi^{-1}(H')$ which is locally compact and $\sigma$-compact and the image of $f$ is a compact set $\mathcal{D} \cap xK$.

We shall show that $f$ is an open mapping. Since every element of $X = \mathcal{D} \cap xK$ can be written as $gxg^{-1}$ for some $g$ in $\pi^{-1}(H')$, the group $\pi^{-1}(H')$ acts on $X$ by conjugation. In fact let $g' \in \pi^{-1}(H')$ then, because $K$ is normal, $g'(gxg^{-1})g'^{-1} \subset g'(xK)g'^{-1} \subset g'xg'^{-1}K \subset xK$. Moreover $\pi^{-1}(H')$ acts transitively on $X$. To see this let $z$ and $z'$ be any two elements in $X$ and write $z = xk$ and $z' = xk'$ ($k, k' \in K$). Clearly there exists an element $h$ in $H$ such that $hkh^{-1} = k'$, and we have $hzh^{-1} = (hkh^{-1}) = xh' = z'$, proving $\pi^{-1}(H')$ acts on $X$ transitively. Thus $f$ is a continuous map of a locally compact and $\sigma$-compact group $\pi^{-1}(H')$ onto a Bair homogeneous $G$-space $X$; $f$ is an open mapping (see the proof of Lemma 3).

Since $f(g) = (g')$ is equivalent to $g^{-1}g' \in H$, the quotient space $\pi^{-1}(H')/H$ is homeomorphic to the compact space $\mathcal{D} \cap xK$, which completes the proof of the theorem.

References

1. V. M. Gluskov, The structure of locally compact groups and Hilbert's fifth problem, Transl. A. M. S. (2) 15 (1960)