ON SOME RELATIONS OF TWO 2-DIMENSIONAL UNIFIED FIELD THEORIES

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1. Introduction

A. 2-dimensional \textit{g-UFT} and \textit{*g-UFT}. In the usual Einstein’s unified field theory (\textit{g-UFT}) the generalized 2-dimensional Riemannian space \(X_2\) referred to a real coordinate system \(x^\nu\) is endowed with a real nonsymmetric tensor \(g_{\lambda \mu}\) which may be split into its symmetric part \(h_{\lambda \mu}\) and skew-symmetric part \(k_{\lambda \mu}^{(*)}:\)

\begin{equation}
(1.1)\ a \quad g_{\lambda \mu} = h_{\lambda \mu} + k_{\lambda \mu},
\end{equation}

where

\begin{equation}
(1.1)\ b \quad \varrho = \text{Det}(g_{\lambda \mu}) \neq 0, \quad \varphi = \text{Det}(h_{\lambda \mu}) \neq 0, \quad \varkappa = \text{Det}(k_{\lambda \mu}) \neq 0.
\end{equation}

The tensor \(h_{\lambda \mu}\) together with \(h^{\lambda \nu}\), uniquely defined by

\begin{equation}
(1.2) \quad h_{\lambda \mu} h^{\lambda \nu} = \delta^\nu_{\mu},
\end{equation}

are used for raising and/or lowering indices in 2-dimensional \textit{g-UFT}.

The differential geometric structure is imposed on \(X_2\) by the tensor \(g_{\lambda \mu}\) by means of a connection \(\Gamma^{\nu}_{\lambda \mu}\) given by the system of Einstein’s equations [3]

\begin{equation}
(1.3) \quad D_w g_{\lambda \mu} = 2 S_{w \nu} g^{\lambda \nu},
\end{equation}

where \(D_w\) denotes the symbol of the covariant derivative with respect to \(\Gamma^{\nu}_{\lambda \mu}\), and \(S_{\lambda \nu} = \Gamma^{\rho}_{[\lambda \nu]}\).

On the other hand, 2-dimensional \textit{*g-UFT} in the same space \(X_2\) referred to a real coordinate system \(x^\nu\) is defined to be based upon the real nonsymmetric tensor \(g^{\lambda \nu}\) defined by

\begin{equation}
(1.5) \quad g_{\lambda \mu} g^{\lambda \nu} = \delta^{\nu}_{\mu}.
\end{equation}

It may also be decomposed into its symmetric part \(h^{\lambda \nu}\) and skew-symmetric part \(k^{\lambda \nu}:

\begin{equation}
(1.6) \quad g^{\lambda \nu} = h^{\lambda \nu} + k^{\lambda \nu}.
\end{equation}

Since \(\text{Det}(h^{\lambda \nu}) \neq 0\), we may define the tensor \(h_{\lambda \mu}\) by

\begin{equation}
(1.7) \quad h_{\lambda \mu} h^{\lambda \nu} = \delta^\nu_{\mu}.
\end{equation}

In the 2-dimensional \textit{*g-UFT} we use both \(h_{\lambda \mu}\) and \(h^{\lambda \nu}\) as tensors for raising and/or lowering indices of all starred tensors defined in \(X_2\) in the usual manner.

(*) Throughout the present paper, Greek indices take the values 1, 2 and follow the summation convention.
We then have, for example,
\[(1.8)a\]
\[*k_{\mu\nu} = *k_{\mu\sigma} *h_{\sigma\nu}\]
so that
\[(1.8)b\]
\[*g_{\mu\nu} = *g_{\mu\sigma} *h_{\sigma\nu}\]

Similarly the differential geometric structure in 2-dimensional \(*g-UFT\) is imposed on \(X_2\) by means of a connection \(\Gamma^\ast_{\mu\nu}\) given by the following system of equations equivalent to (1.3):
\[(1.9)\]
\[D_\alpha *g^{\mu\nu} = -2S_{\alpha\beta\gamma} *g^{\beta\gamma}\]

Using the following densities and scalars, we define the following:
\[(1.10)\]
\[*q=\text{Det}(g_{\mu\nu}), *\mathcal{K}=\text{Det}(h_{\mu\nu}), *g=g/\mathcal{J}, k=\mathcal{K}/\mathcal{J}, *k=\mathcal{K}/\mathcal{J},\]

Chung [1] proved that two unified tensor fields \(g_{\mu\nu}\) and \(*g_{\mu\nu}\) are related by
\[(1.11)a\]
\[*h^{\mu\nu} = \frac{1}{g} h^{\mu\nu}, *k^{\mu\nu} = \frac{1}{g} k^{\mu\nu}\]
\[(1.11)b\]
\[*h_{\mu\nu} = g h_{\mu\nu}, *k_{\mu\nu} = g k_{\mu\nu}\]

and that
\[(1.12)\]
\[g=1+k, *g=1+*k\]

In both 2-dimensional unified field theories, it is obvious that there exists only the first class since
\[\mathcal{K}=(k_{12})^2>0, *\mathcal{K}=*(k_{12})^2>0.\]

B. Purpose. The purpose of the present paper is to derive some relations of 2-dimensional \(g-UFT\) and \(*g-UFT\) other than (1.11) and (1.12). These results are used to investigate the relationship between two different expressions of torsion tensor \(S_{\mu\nu\rho}\), which lead to a solution of (1.3) and (1.9) in Einstein's 2-dimensional unified field theories.

2. Some relations of 2-dimensional \(g-UFT\) and \(*g-UFT\)

In this Section, we derive several relations of two 2-dimensional unified field theories and obtain a simple expression for the torsion tensor in 2-dimensional \(*g-UFT\).

**Theorem (2.1).** The scalars defined in (1.10) are related by
\[(2.1)\]
\[*g=g, *k=k\]

**Proof.** Putting \(*g=\text{Det}(g^{\mu\nu})\), we have
\[(2.2)\]
\[*q*=\*\mathcal{K}=g^2\mathcal{J}, *\mathcal{K}=g^2\mathcal{J}\]
which may be obtained from (1.5), (1.8)a, and (1.11)b. The relations
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(2.1) are results of (1.10) and (2.2). An alternative proof of \( *k = k \) is obtained from (1.12) using \( *g = g \).

**Theorem (2.2).** We have

\[
* \left[ \frac{\alpha}{\lambda \alpha} \right] = \left[ \frac{\alpha}{\lambda \alpha} \right] + \frac{g_{\lambda \alpha}}{g} \left( \frac{\partial g}{\partial x^\lambda} \right),
\]

where \( \left[ \frac{\gamma}{\lambda \mu} \right] \) and \( \left[ \frac{\nu}{\lambda \mu} \right] \) are the Christoffel symbols of the second kind formed with respect to \( h_{\lambda \mu} \) and \( *h_{\lambda \mu} \), respectively.

*Proof.* In virtue of (1.11)b, two Christoffel symbols are related by

\[
* \left[ \frac{\nu}{\lambda \mu} \right] = \left[ \frac{\nu}{\lambda \mu} \right] + \frac{1}{4g} (g_{\nu}, \partial_\nu a + g_{\nu}, \partial_{\nu} a - g_{\nu}, \partial_{\nu} a). \]

Hence

\[
* \left[ \frac{\alpha}{\lambda \alpha} \right] = \left[ \frac{\alpha}{\lambda \alpha} \right] + \frac{1}{2g} (g_{\alpha}, \partial_\alpha a + g_{\alpha}, \partial_{\alpha} a - g_{\alpha}, \partial_{\alpha} a),
\]

which proves (2.3).

**Theorem (2.3).** We have

\[
* V_{\nu} k_{\alpha \mu} = g V_{\nu} k_{\alpha \mu},
\]

where \( V_\nu \) and \( *V_\nu \) are the symbolic vector of the covariant derivative with respect to \( \left[ \frac{\nu}{\lambda \mu} \right] \) and \( \left[ \frac{\nu}{\lambda \mu} \right] \), respectively.

*Proof.* Since \( k_{\alpha \mu} \) is skew-symmetric, it suffices to show that \( *V_\nu k_{12} = g V_\nu k_{12} \). This result follows in the following way, using (1.11)b and (2.3):

\[
* V_{\nu} * k_{12} = \partial_\nu * k_{12} - \left[ \frac{\beta}{\nu 1} \right] * k_{12} - \left[ \frac{\beta}{\nu 2} \right] * k_{12} = \partial_\nu (g k_{12}) - g k_{12} = g V_{\nu} k_{12}.
\]

**Remark.** Chung [2] proved that in 2-dimensional \( g-UFT \) the torsion tensor \( S_{\alpha \mu \nu} \) satisfying Einstein’s equations (1.3) is given by

\[
S_{\alpha \mu \nu} = \frac{1}{g} V_\nu k_{\alpha \mu}.
\]

In virtue of (2.1) and (2.5), we see that in 2-dimensional \( *g-UFT \) the same torsion tensor \( S_{\alpha \mu \nu} \) (satisfying (1.9)) may be given by a simple expression

\[
S_{\alpha \mu \nu} = \frac{1}{g^2} * V_\nu k_{\alpha \mu}.
\]
References

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