HONAM MATHEMATICAL JOURNAL
Volume 2, Number 1, July 1980

SOME RELATIONS WITH COHOMOLOGY GROUPS

HEE-JIN LEE

ABSTRACT

本 論文은 Cohomotopy群과 Cohomolopy群 사이의 몇가지 關係를 究明한 것이다
論文의 主部分으로서

i) 어떤 條件下에서 $\pi^n(X,A) \cong H^n(X,A)$ (Theorem 2, Corollary 3)

ii) 어떤 條件下에서 $\pi^n(X,A)=0 \Leftrightarrow H^n(X,A)=0$ (Theorem 4)

iii) 어떤 條件下에서 $\pi^n(X) \cong \pi^n(A) \Leftrightarrow H^n(X) \cong H^n(A)$ (Corollary 5)

가 成立함을 証明하였다.

1. INTRODUCTION

The homotopy theory has arised from extension problems and lifting problems in topology. After W. Hurewicz's work(1935), the homotopy theory has been studied by many mathematicians, for example, by Arens [1], Curtis [2], Hu [6]~[8] and Spanier [15], etc. In particular, at the first time Spanier constructed the cohomotopy theory as a dual concepts of the homotopy theory. The special study with respect to the cohomotopy theory is found in Hu's work [7] and [8].

The purpose of this paper is to prove some relations between cohomotopy and cohomology groups. We shall prove that

i) Under some conditions $\pi^n(X,A) \cong H^n(X,A)$ (Theorem 2, Corollany 3),

ii) So under some conditions $\pi^n(X,A)=0 \Leftrightarrow H^n(X,A)=0$ (Theorem 4) and

iii) Under some conditions $\pi^n(X) \cong \pi^n(A) \Leftrightarrow H^n(X) \cong H^n(A)$ (Corollary 5).

2. Some Relations with Cohomology Groups

For a topological pair (X,A)

$$h^m:\ \pi^m(X,A) \longrightarrow H^m(X,A);\ \pi_m(S^m,s_o))$$

is defined as follows. Since S^m is a finite cell complex which is $(m\text{-}1)$-connect ed, there is a unique characteristic element x_m of S^m. Take an element $\alpha \in \pi^m(X,A)$ with a representation ϕ: $(X,A)\rightarrow(S^m, s_o)$. Then there is a group homomorphism

$$\phi^*: H^m(S^m,s_o;\ \pi_m(S^m,s_o))\longrightarrow H^m(X,A;\ \pi_m(S^m,s_o)).$$

Accordingly, $\phi^*(\chi_m)$ is an element of $H^m(X,A;\pi_m(S^m,s_o))$. We define

$$\begin{array}{lll} & h^m: \pi^m(X,A)\longrightarrow H^m(X,A;\ \pi_m(S^m,s_o)), \ m>0, & \\ & \quad\ \ \Cup & \\ \text{by} & \quad\ \ \alpha \quad \longmapsto h^m(\alpha)=\phi^*(\chi_m) & \\ & \quad\ \ \Cup & \\ & \quad\ \ \phi & \end{array}$$

where ϕ is a representation of α.

As is well-known, the natural homomorphism h^*: $\pi_m(X,A,x_o)\rightarrow H_m(X,A)$ is defined by

$$\begin{array}{lll} h_m: & \pi_m(X,A,x_o)\longrightarrow & H_m(X,A) \\ & \quad\ \Cup & \quad \Cup \\ & \quad\ \alpha & h_m(\alpha)=\phi_*(\xi_n) \\ & \quad\ \Cup & \\ & \quad\ \phi & \end{array}$$

where ϕ is a representation of α and ξ_n the generator of $H_m(E^m, S^{m-1})\cong Z$ (all integers).

PROPOSITION 1. For the homomorphism h_m: $\pi_m(S^m,s_o)\rightarrow H_m(S^m,s_o)$, h_m^{-1} completely determines the characteristic element χ_m of S^m.

PROOF. By Hurewicz's theorem [8] h_m is an isomorphism. If $i_*(\xi_n)$ is a generator $h^{-1}_m(i_*(\xi_n))\in\pi_m(X,A,x_o)$ is the homotopy class of the identity map i: $S^m\rightarrow S^m$. Hence $x^m(h^{-1}_m(i^*(\xi_n)))=\chi_m$ is a generator of $H^m(S^m,s_o)\cong Z$.

THEOREM 2. For a positive integer m, let (X,A) be an $(m+1)$—coconnected cellular pair. Then h^m: $\pi^m(X,A)\rightarrow H^m(X,A)$ is an one-to-one and onto correspondence.

PROOF. We shall put

W=the set of all maps f: $X\rightarrow S^m$ with $f(A)=s_o\in S^m$.

Since (X,A) is $(m+1)$-coconnected we have

$$H^{r+1}(X,A;\ \pi_r(S^m,s_o))=H^r(X,A;\ \pi_r(S^m,s_o))=0$$

for all $r>m$. Since S^m is r-simple of all $r\geq 1$. We have that the homotopy classes of the maps W relative to A are in an one-to-one correspondence with the elements of the group

$$H^m(X,A;\ \pi_m(S^m,s_o))=H^m(X,A;\ \pi_m(S^m))=H^m(X,A)$$

by $g \mapsto \omega^m(f,g)$, where f is a given map in W. Since the homotopy classes of the maps W relative to A is just equal to $\pi^m(X,A)$, $\pi^m(X,A)$ and $H^m(X,A)$ are in an one-to-one and onto correspondence as sets. Recall that

$$\begin{array}{rcl} h^m:\ \pi^m(X,A) & \longrightarrow & H^m(X,A) \\ \Downarrow & & \Downarrow \\ \alpha & \longmapsto & h_m(\alpha)=\phi^*(\chi_m) \end{array}$$

where ϕ is a representation of α and

$$\chi_m=\kappa^m(1_{S^m})=\omega^m(1_{S^m},\theta)(\theta(S^m)=s_0)$$

Since $\phi^*(\chi_m)=\kappa^m(\phi)=\omega^m(\phi,\theta)$, if we fix the map θ in W we know that h^m coincides with the correspondence $g \mapsto \omega^m(f,g)$.

COROLLARY 3. Under the above situation, if (X,A) is $(2m\text{-}1)$-coconnected,

$$h^m:\ \pi^m(X,A)\longrightarrow H^m(X,A)$$

is a group isomorphism.

PROOF. Since $\pi^m(X,A)$ is an abelian group and h^m is an one-to-one and onto correspondence, it suffices to prove that h^m is a group homomorphism.

Let p_i: $S^m\times S^m\to S^m$ $(i=1,2)$ be the canonical projection, i.e., $p_i(y_1,y_2)=y_i(i=1,2$ and $(y_1,y_2)\epsilon S^m\times S^m)$, and let j: $S^m \vee S^m\to S^m$ be defined by $j(s,s_0)=s=j(s_0,s)$.
Let k: $S_m \vee S^m\to S^m\times S^m$ qe the inclusion map.
Then we have the following induced maps.

$$p^*_i:\ H^m(S^m,s_0)\longrightarrow H^m(S^m\times S^m,s_0\times s_0)$$
$$j^*:\ H^m(S^m,s_0)\longrightarrow H^m(S^m\vee S^m,s_0\times s_0)$$
$$k^*:\ H^m(S^m\times S^m,s_0\times s_0)\longrightarrow H^m(S^m\vee S^m,s_0\times s_0)$$

Then it is easy to prove that

$$j^*(\chi_m)=k^*p^*_1(\chi_m)+k^*p^*_2(\chi_m) \qquad (※)$$

Take two maps $\phi,\Psi:(X,A)\longrightarrow(S^m,s_0.)$ Then, since

$$\pi(X,A;(S^m\vee S^m,s_0\times s_0))\simeq\pi(X,A;\ S^m\times S^m,s_0\times s_0)$$

there exists a map g: $(X,A)\longrightarrow(S^m\vee S^m,s_0\times s_0)$ such that $\phi\times\Psi\simeq g$ *rel* A.
Then we have $\phi\simeq p_1kg$ *rel* A, $\Psi\simeq p_2kg$ *rel* A. Therefore

$$h^m([\phi])+h^m([\Psi])=\phi^*(\chi_m)+\Psi^*(\chi_m)$$
$$=g^*k^*p^*_1(\chi_m)+g^*k^*p^*_2(\chi_m)$$
$$=g^*j^*(\chi_m) \quad (\text{by } (※))$$

In this case, $[jg]=[\phi]+[\Psi]$, and thus

$$h^m([\phi]+[\Psi])=(jg)^*(\chi_m)=g^*j^*(\chi_m).$$

Therefore we have that $h^m([\phi]+[\Psi])=h^m([\phi])+h^m([\Psi])$, which means that h^m is a group homomorphism.

THEOREM 4. Let (X,A) be a triangulable pair such that

$$H^n(A)=0=H^n(X)$$

for all $n\geq 2m-1$, where m is a given positive integer. Then $H^n(X,A)=0$ if and only if $\pi^n(X,A)=0$ for all $n>m$. In this case, $\pi^m(X,A)=H^m(X,A)$.

PROOF. By the universal coefficient theorem([10],[15]):

$$0\longrightarrow H^q(X)\otimes G\longrightarrow H^q(X;G)\longrightarrow H^{q+1}(X)*G\longrightarrow 0 \quad (\text{exact}),$$

where G is an arbitrary coefficient group, it follows that X is n-coconnected if and only if $H^q(X)=0$ for all $q\geq n$. Assume that $H^n(X,A)=0$ for $n>m$, then it follows that (X,A) is $(m+1)$-coconnected.
By Corollary 3 the homomorphism

$$h^m\colon \pi^m(X,A)\longrightarrow H^m(X,A)$$

is an isomorphism (Note that our hypothesis implies that $\pi^m(X,A)$ is an abelian group). For some $n>m$, since (X,A) is $(n+1)$-coconnected $h^n\colon \pi^n(X,A)\cong H^n(X,A)=0$.

Conversely, we assume that $\pi^n(X,A)=0$ for all $n>m$. This implies that there are no any r-simplex $(r\geq n)$ in X-A. Thus $H^n(X,A)=0$.

COROLLARY 5. Under the above situation

$$i^*\colon H^n(X)=H^n(A)\Leftrightarrow i^*\colon \pi^n(X)=\pi^n(A)$$

for all $n>m$, where $i\colon A\rightarrow X$ is the inclusion map.

PROOF. Consider the following commutative diagarm about the cohomotopy long exact sequence and the cohomology long exact sequence ([10],[15]) with respect to $A\hookrightarrow X\hookrightarrow (X,A)$:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \pi^n(X,A) & \longrightarrow & \pi^n(X) & \xrightarrow{i^*} & \pi^n(A) & \longrightarrow & \pi^{n+1}(X,A) & \longrightarrow \cdots & \text{(exact)} \\
h^n \downarrow & \copyright & h^n \downarrow & \copyright & h^n \downarrow & \copyright & h^{n+1} \downarrow & & \\
\cdots \longrightarrow H^n(X,A) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A) & \longrightarrow & H^{n+1}(X,A) & \longrightarrow \cdots & \text{(exact)}
\end{array}$$

In this diagram, we have the following ($n>m$):

$$H^n(X)\overset{i^*}{\cong}H^n(A) \Leftarrow\Rightarrow H^n(X,A)=O \Leftarrow\Rightarrow \pi^n(X,A)=O \Leftarrow\Rightarrow \pi^n(X)\overset{i^*}{\cong}\pi^n(A)$$

by Theorem 4.

ACKNOWLEDGMENT

I wish to thank my chairman, academic advisor Dr. KEEAN LEE for his constant encouragement and valuable comments in the making of this paper.

REFERENCES

1. Arens, R., *A Topology for Spaces of Transformations*, Ann. of Math., 47 (1946), 480∼495.
2. Curtis, M.L., *The Covering Homotopy Theorem*, Proc. Amer. Math. Soc., 7 (1956), 682∼984.
3. Dugundji, J., *An Extension of Tietze's Theorem*, Pacific J. Math., 1 (1951), 353∼367.
4. Eilenberg, S. and Maclane, S., *Relations between Homology and Homotopy Groups of spaces*, Ann. of Math., 46 (1945), 480∼509. II, Ann. of Math., 51 (1950), 514∼533.
5. Eilenberg, S. and Steenrod, N.D., *Function of Algebraic Topology*, Princeton Univ. Press (1952).
6. Hu, S.T., *Extension and Classification of Maps*, Osaka Math. J., 2 (1950), 165∼209.
7. ________, *Cohomolgy and Deformation Retacts*, Proc. London Math. Soc.,(2)53 (1951), 191∼219.
8. ________, *Homotopy Theory*, Academic Press (1959).
9. Huebsch, W., *On the Covering Homotopy Theorem*, Ann. of Math., 61 (1955), 555∼563.
10. Mac Lane, S., *Homotopy*, Springer Verlag (1963).
11. Moore, J.C., *Some Applications of Homology Theory to Homotopy Problems*, Ann. of Math., 58 (1953), 325 ∼350.
12. Munkres, *Topology*, Prentice-Hall Inc. (1975).
13. Olum, P., *Obstructions to Extensions and Homotopies*, Ann. of Math., 52 (1950), 1∼50.
14. Serre, J.P., *Groups d'homotopie et classes de Groupe Abelians*, Ann. of Math., 58 (1953), 258∼294.
15. Spanier, E.H., *Algebraic Topology*, McGraw-Hill Book Company 1966.

(Jeonbug National University)