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ON VECTOR-VALUED INTEGRATION

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ABSTRACT

Vector値 函數의 Vector値 積分論에는 많은 硏究가 되어 왔으나, 本 論文에서는 그 가운데 Bochner 積分論에 對해서 硏究하고 Bochner 不定積分에 依한 Vector値 測度의 表現에 關한 Radon-Nikodym 定理에 對해서 硏究하였다.

Introduction

As a generalization of the Lebesgue integration, we may integrate the functions that are defined on some measure space and whose values lie in some vector space. Since many approaches are given to this problem, there are various kinds of vector-valued integrations. Among such integrations, in this paper, we study the Bochner integration. First, we give a representation theorem for the Bochner integrable functions and then establish a Radon-Nikodym theorem for the vector-valued measures, which can be the Bochner indefinite integrals.

1. The Bochner Integral

Let X be a Banach space and $(S, \mathfrak{M}, \mu)$ be a measure space where S is a measurable subset of the real numbers $\boldsymbol{R}$ with $\mu(S)<\infty$, $\mathfrak{M}$ is the σ-algebra of measurable subsets of S and μ is a measure on $\mathfrak{M}$. A function $x: S \to X$ is called a vector-valued function.

1-1. DEFINITION Let $x: S \to X$ be a vector-valued function.

(1) x is said to be weakly measurable if $f(x)$ is measurable for every $f \in X^*$, where X^* is the conjugate space of X.

(2) x is called a simple function if it is a constant vector ($\neq \theta$) on each of a finite number of disjoint sets $A_i \in \mathfrak{M}$, $i=1,2,\cdots,n$, and $x(s)=\theta$ for $s \in S - \bigcup_{i=1}^{n} A_i$, where θ is the origin of X.

(3) x is said to be measurable if there is a sequence $\{x_n\}$ of simple functions such that

$$\lim_{n\to\infty} \left\| x_n(s)-x(s) \right\| = 0 \quad a.e. \text{ in } S.$$

(4) A space X is said to be separable if there exists a countable dense subset of X.

(5) x is said to be separably valued if there exists a separable subspace Y of X and a set $S' \subset S$ such that $\mu(S')=\mu(S)$ and $x(S') \subset Y$.

1-2. LEMMA Let X be a separable Banach space and $U^* = \{f \in X^*: \| f \| \leq 1\}$ be the unit sphere of X^*. Then there exists a sequence $\{f_n\}$ in U^* such that for any $f_0 \in U^*$, we can choose a subsequence $\{f_{n(i)}\}$ of $\{f_n\}$ satisfying that $\lim_{i\to\infty} f_{n(i)}(x)=f_0(x)$ at every $x \in X$.

PROOF Let $\{x_n\}$ be a dense sequence in X and $l^2(n)$ be n-dimensional Hilbert space of vectors $(\xi_1, \cdots, \xi_n)$ normed by $\| (\xi_1, \cdots, \xi_n) \| = \left(\sum_{i=1}^{n} | \xi_i |^2 \right)^{\frac{1}{2}}$. For fixed $n \in N$, consider a mapping $\varphi_n : U^* \to l^2(n)$ with $\varphi_n(f)=(f(x_1), \cdots, f(x_n))$. Since the space $l^2(n)$ is separable, there is a sequence $\{f_{n,k} : k=1,2,\cdots\}$ in U^* such that $\{\varphi_n(f_{n,k}) : k=1,2,\cdots\}$ is dense in $\varphi_n(U^*)$. Thus for any $f_0 \in U^*$, we can choose a subsequence $\{f_{n,k(n)} : n=1,2,\cdots\}$ of $\{f_{n,k} : k=1,2,\cdots\}$ such that

$$| f_{n,k(n)}(x_i) - f_0(x_i) | < \frac{1}{n} \qquad (i=1,2,\cdots,n).$$

Hence $\lim_{n\to\infty} f_{n,k(n)}(x_i)=f_0(x_i)$ for all $x_i \in \{x_n\}$. For given $x \in X$ and $\varepsilon > 0$, there is an $x_N \in \{x_n\}$ with $\| x - x_N \| < \frac{\varepsilon}{2}$, and

$$\begin{aligned} &| f_{n,k(n)}(x) - f_0(x) | \\ &\leq | f_{n,k(n)}(x) - f_{n,k(n)}(x_N) | + | f_{n,k(n)}(x_N) - f_0(x_N) | + | f_0(x_N) - f_0(x) | \\ &\leq \| f_{n,k(n)} \| \, \| x - x_N \| + | f_{n,k(n)}(x_N) - f_0(x_N) | + \| f_0 \| \, \| x - x_N \| \\ &\leq \varepsilon + \frac{1}{n} \end{aligned}$$

Thus $\lim_{n\to\infty} f_{n,k(n)}(x)=f_0(x)$ for every $x \in X$.

1-3. THEOREM (B.J. Pettis) A vector-valued function x is measurable if and only if it is weakly measurable and separably valued.

PROOF Suppose that x is measurable, then there is a sequence $\{x_n\}$ of simple functions such that

$$\lim_{n\to\infty} \| x_n(s) - x(s) \| = 0 \quad a.e. \text{ in } S.$$

Since each x_n is weakly measurable, x is weakly measurable. Let $A_n = \{x_n(s) : s \in S\}$ and A be the set of all finite linear combinations of elements in $\bigcup A_n$. Then the closure $\bar{A}$

is a separable subspace of X. Hence $x(s) \in \bar{A}$ for almost all $s \in S$.

Suppose that x is weakly measurable and without loss of generality we may assume that all the values of x lie in a separable space $Y \subset X$. We first prove that $\| x(s) \|$ is measurable. For any $r \in \boldsymbol{R}$, put

$$A = \{s \in S : \| x(s) \| \leq r\}$$
$$A_f = \{s \in S : | f(x(s)) | \leq r\}, \text{ where } f \in U^*.$$

We claim that $A = \bigcap \{A_{f_i} : i = 1, 2, \cdots\}$ for some sequence $\{f_i\} \subset U^*$. If $s \in A$, then $\| x(s) \| \leq r$, thus for any $f \in U^*$, $| f(x((s)) | \leq \| x(s) \| \leq r$ and this implies $A \subset \bigcap \{A_f : f \in U^*\}$. On the other hand, for fixed $s \in S$, by the Hahn-Banach theorem there exists $f_0 \in X^*$ with $\| f_0 \| \leq 1$ and $f_0(x(s)) = \| x(s) \|$. Thus $\| x(s) \| = | f_0(x(s)) | \leq r$, hence $\bigcap \{A_f : f \in U^*\} \subset A$ and therefore $A = \bigcap \{A_f : f \in U^*\}$. But by 1.2 LEMMA, there is a sequence $\{f_i\}$ in U^* such that for any $f \in U^*$, we can choose a subsequence $\{f_{i(n)}\}$ of $\{f_i\}$ satisfying

$$\lim_{n \to \infty} f_{i(n)}(x) = f(x) \text{ at every } x \in X.$$

Thus if $s \in \bigcap \{A_{f_i} : i = 1, 2, \cdots\}$, then

$$| f(x(s)) | = \lim_{n \to \infty} | f_{i(n)}(x(s)) | \leq r$$

for every $f \in U^*$, hence

$$A = \bigcap \{A_f : f \in U^*\} = \bigcap \{A_{f_i} : i = 1, 2, \cdots\}.$$

Since x is weakly measurable, $A \in \mathfrak{A}$ and thus $\| x(s) \|$ is measurable. By the separability of the space Y, there is a countable dense subset $\{x_j : j = 1, 2, \cdots\}$ of Y.
For each $n \in \boldsymbol{N}$, let

$$B_{jn}(x_j) = \{x \in Y : \| x - x_j \| < \tfrac{1}{n}\},$$

then $B_{jn}(x_j)$, $j = 1, 2, \cdots$, are open balls with $Y = \bigcup_{j=1}^{\infty} B_{jn}(x_j)$ and the sets $S'_{jn} = \{s \in S : x(s) \in B_{jn}(x_j)\}$, $j = 1, 2, \cdots$, are measurable as proved above and $S = \bigcup_{j=1}^{\infty} S'_{jn}$. Let $S_{1n} = S'_{1n}$ and $S_{in} = S'_{in} - \bigcup_{j=1}^{i-1} S'_{jn}$ for $i = 2, 3, \cdots$, then family $\{S_{in} : i = 1, 2, \cdots,\}$ is a class of disjoint measurable sets with $S = \bigcup_{i=1}^{\infty} S_{in}$. If we set $E_{N,n} = \bigcup_{i=N+1}^{\infty} S_{in}$ for $N \in \boldsymbol{N}$, then for any $s \in S$ there exists a number $k \in \boldsymbol{N}$ such that $s \in S_{kn}$ and $N > k$ implies $s \notin E_{N,n}$, hence

$\bigcap_{N=1}^{\infty} E_{N,n}=\phi$. And since $E_{N+1,n}\subset E_{N,n}$, we get $\lim_{N\to\infty}\mu(E_{N,n})=\mu(\bigcap_{N=1}^{\infty} E_{N,n})=0$. Thus for any $m\in N$, we can find a number $k(n,m)\in N$ such that $\mu(E_{k(n,m),n})<\frac{1}{2^n}\ \frac{1}{m}$ and $k(n,m_1)\leq k(n,m_2)$ if $m_1<m_2$. Let $E_m=\bigcup_{n=1}^{\infty} E_{k(n,m),n}$, then $\mu(E_m)\leq\sum_{n=1}^{\infty}\frac{1}{2^n}\ \frac{1}{m}=\frac{1}{m}$ and $E_{m_1}\supset E_{m_2}$ if $m_1<m_2$. Hence $\mu(\bigcap_{m=1}^{\infty} E_m)=\lim_{m\to\infty}\mu(E_m)=0$. We set $x'_n(s)=x_i$ if $s\in S_{in}$ and set

$$x_m(s)=\begin{cases}x'_m(s) & \text{if } s\in S-E_m\\ \theta & \text{if } s\in E_m\end{cases}$$

then $\{x_m:m=1,2,\cdots\}$ is a sequence of simple functions satisfying that

$$\lim_{m\to\infty}\|x_m(s)-x(s)\|=0 \quad a.e. \text{ in } S.$$

For let $\varepsilon>0$ and $s\in S-\bigcap_{m=1}^{\infty} E_m$ be arbitrarily given, then since $s\in S-\bigcap_{m=1}^{\infty} E_m=\bigcup_{m=1}^{\infty}(S-E_m)$, there is a number $m_0\in N$ such that $s\notin E_{m_0}\supset E_{m_1}$ for any $m_1\geq m_0$. Take $l=max(\frac{1}{\varepsilon},m_0)$ and if $m\geq l$, then $s\notin E_m$ and

$$\|x_m(s)-x(s)\|\leq\|x_m(s)-x'_m(s)\|+\|x'_m(s)-x(s)\|<\frac{1}{m}<\varepsilon.$$

Hence we conclude that x is measurable.

1-4 COROLLARY If $\{x_n\}$ is a sequence of measurable functions converging weakly $a.e.$ to x, then x is measurable.

PROOF Since x_n is measurable, by the theorem x_n is separably valued for every $n\in N$. If Y_n is a separable closed subspace containing almost all the values of x_n, let Y be the closed linear hull of a countable set dense in $\bigcup Y_n$. Then Y is separable. If $x_n(s)$ converges to $x(s)$ weakly at $s\in S$, then some sequence of linear combinations of the elements in $\{x_n(s):n=1,2\cdots\}$ converges to $x(s)$ (*Functional Analysis*, K.Yosida, p.120) $x(s)$ is in Y. This holds for almost all $s\in S$ and x is weakly measurable, because the limit function of the measurable functions is measurable, hence by the theorem x is measurable.

For a simple function $x(s)=\sum_{i=1}^{n} x_i\chi_{A_i}(s)$, where χ_{A_i} is the characteristic function of $A_i\in\mathfrak{M}$ and A_i are disjoint, we define the Bochner integral of x over $A\in\mathfrak{M}$

$$\int_A x\, d\mu = \sum_{i=1}^{n} x_i \mu(A \cap A_i).$$

Then the following properties hold from the definition:

(1) Linearity: For simple functions x_1, x_2 and for $a, b \in \mathbf{R}$

$$\int_A (ax_1 + bx_2) d\mu = a\int_A x_1 d\mu + b\int_A x_2 d\mu$$

(2) $\left\| \int_A x\, d\mu \right\| \leq \int_A \| x \| d\mu$

Generalizing the above fact, we have:

1-5. DEFINITION Let X be a separable Banach space and $(S, \mathfrak{A}, \mu)$ be a measure space as before. A vector-valued function $x: S \rightarrow X$ is said to be Bochner integrable if there exists a sequence $\{x_n\}$ of simple functions satisfying the conditions

(1) $\lim_{n\to\infty} \| x(s) - x_n(s) \| = 0 \quad a.e.$ in S

(2) $\lim_{n\to\infty} \int_S \| x(s) - x_n(s) \| d\mu = 0.$

When $x: S \rightarrow X$ is Bochner integrable, the Bochner integral of x over $A \in \mathfrak{A}$ is defined by

$$\int_A x\, d\mu = \lim_{n\to\infty} \int_A x_n d\mu,$$

where $\{x_n\}$ is a sequence of simple functions satisfying the above conditions (1) and (2).

1-6. REMARK The condition (2) has a sense since x and $\| x(s) - x_n(s) \|$ are measurable as shown in the proof of 1.3 THEOREM.

Existence: From the condition (2) and the inequality

$$\begin{aligned} \left\| \int_A x_n d\mu - \int_A x_m d\mu \right\| &= \left\| \int_A (x_n - x_m) d\mu \right\| \\ &\leq \int_A \left\| x_n - x_m \right\| d\mu \\ &\leq \int_A \left\| x_n - x \right\| d\mu + \int_A \left\| x - x_m \right\| d\mu, \end{aligned}$$

$\left\{ \int_A x_n d\mu \right\}$ is a Cauchy sequence in the Banach space X. Thus $\lim_{n\to\infty} \int_A x_n d\mu$ exists in X.

Uniqueness: Since every Banach space is Hausdorff space, this limit is unique and is independent of the approximating sequences satisiying the conditions (1) and (2).

1-7. THEOREM (S.Bochner) A measurable function x is Bochner integrable if and only if $\| x(s) \|$ is integrable.

PROOF Let $\{x_n\}$ be a sequence of simple functions such that x_n converges to x a.e. Suppose that x is Bochner integrable. Since $\|x\| \leq \|x_n\| + \|x-x_n\|$,

$$\int_A \|x\| d\mu \leq \int_A \|x_n\| d\mu + \int_A \|x-x_n\| d\mu.$$

Hence by the integrability of $\|x_n\|$ and the condition (2), $\|x\|$ is integrable. Moreover, since

$$\int_A \Big| \|x_n\| - \|x_m\| \Big| d\mu \leq \int_A \|x_n - x_m\| d\mu,$$

by the condition (2), $\lim_{n\to\infty}\int_A \|x_n\| d\mu$ exists so that we have

$$\int_A \|x\| d\mu \leq \lim_{n\to\infty}\int_A \|x_n\| d\mu.$$

Conversely if $\|x\|$ is integrable, for each $n \in N$, put

$$y_n(s) = \begin{cases} x_n(s) & \text{if } \|x_n(s)\| \leq \|x(s)\| (1+\frac{1}{n}) \\ \theta & \text{if } \|x_n(s)\| > \|x(s)\| (1+\frac{1}{n}). \end{cases}$$

Then $\{y_n\}$ is a sequence of the simple functions satisfying

$$\|y_n(s)\| \leq \|x(s)\| (1+\frac{1}{n})$$

$$\lim_{n\to\infty} \|x(s) - y_n(s)\| = 0 \qquad a.e. \text{ in } S.$$

Since $\|x\|$ is integrable and

$$\|x-y_n\| \leq \|x\| + \|y_n\| \leq \|x\| + \|x\| (1+\frac{1}{n}) \leq 2\|x\| (1+\frac{1}{n}),$$

by the Fatou's lemma

$$\overline{\lim_{n\to\infty}}\int_S \|x-y_n\| d\mu \leq \int_S \overline{\lim_{n\to\infty}} \|x-y_n\| d\mu = 0.$$

Hence $\lim_{n\to\infty}\int_S \|x(s)-y_n(s)\| d\mu = 0$ and x is Bochner integrable.

1-8. COROLLARY Let X, Y be Banach spaces and $T: X \to Y$ be a bounded linear mapping. If $x: S \to X$ is Bochner integrable, Tx is Bochner integrable and

$$\int_A Tx\, d\mu = T(\int_A x\, d\mu).$$

PROOF Let $\{y_n\}$ be a sequence of the simple functions satisfying that $\| y_n(s) \| \leq \| x(s) \| (1+\frac{1}{n})$ and y_n converges to x a.e. in S. Then since T is linear

$$\int_A Ty_n(s)\, d\mu = T\int_A y_n(s)\, d\mu$$

and by the continuity of T,

$$\| Ty_n(s) \| \leq \| T \| \; \| y_n(s) \| \leq \| T \| \; \| x(s) \| (1+\frac{1}{n})$$

and $\lim_{n\to\infty} Ty_n(s) = Tx(s)$ a.e. Thus $\| Tx(s) \|$ is integrable and by the theorem $Tx(s)$ is Bochner integrable. Furthermore

$$\int_A Tx(s)d\mu = \lim_{n\to\infty}\int_A Ty_n(s)d\mu = \lim_{n\to\infty} T\int_A y_n(s)d\mu = T\int_A x(s)d\mu.$$

1-9. THEOREM If x is Bochner integrable and $A\in\mathfrak{M}$ is a support of x with $0<\mu(A)<\infty$, then $\frac{1}{\mu(A)}\int_S x\, d\mu$ belongs to the closed convex hull of $x(A)$.

PROOF Let C be the closed convex hull of $x(A)$ and suppose that $\frac{1}{\mu(A)}\int_S x\, d\mu \notin C$. Then, by the Hahn-Banach theorem, there is $f\in X^*$ such that $f(\frac{1}{\mu(A)}\int_S x\, d\mu)$ is at a positive distance from $f(C)$. But $f(\frac{1}{\mu(A)}\int_S x\, d\mu) = \frac{1}{\mu(A)}\int_S f(x)d\mu$ and $f(C)$ is a convex set containing $f(x(A))$, thus $\frac{1}{\mu(A)}\int_S f(x)d\mu$ is at a positive distance from the convex hull of $f(x(A))$, which is impossible.

2. Radon-Nikodym Theorem for the Bochner Indefinite Integral

2-1. DEFINITION Let X be a Banach space and $(S, \mathfrak{M}, \mu)$ be measure space as before.

(1) A countably additive function $\nu: \mathfrak{M}\to X$ is called a vector-valued measure on $\mathfrak{M}$.

(2) If x is Bochner integrable, the Bochner indefinite integral $x\mu$ is defined by

$$x\mu(A) = \int_A x\, d\mu \text{ for } A\in\mathfrak{M}.$$

(3) If ν is a vector-valued measure on $\mathfrak{M}$, then the variation V_ν of ν is defined by

$$V_\nu(A) = \sup\ \{\Sigma \| \nu(A_n) \| : \{A_n\} \text{ is disjoint sequence in } \mathfrak{M} \text{ with } A_n\subset A\}.$$

2-2. REMARK Followings are hold from the definition:

(1) The Bochner indefinite integral $x\mu$ is a vector-valued measure on $\mathfrak{M}$.

(2) The variation V_ν is a measure on $\mathfrak{M}$.

(3) If $\nu = x\mu$, then $V_\nu = \|x\|\mu$ and V_ν is finite.

(4) If $\nu = x\mu$, then $\nu = \frac{x}{\|x\|}\|x\|\mu$, where $\frac{x}{\|x\|}$ is defined to be θ when $x = \theta$.

(5) If $\nu = y \cdot V_\nu$ for some vector-valued function y and if $V_\nu = \varphi \cdot u$ for some scalar function φ, then $\nu = (y\varphi) \cdot u$.

PROOF (3) If $x\mu$ is a Bochner indefinite integral, then x is measurable, thus there exists a sequence $\{x_n\}$ of simple functions converging to x a.e. in S. If x is a constant vector, then

$$\|x\mu(A)\| = \left\|\int_A x \, d\mu\right\| = \|x\|\mu(A).$$

Hence for simple function x_n, we have

$$\Sigma\|x_n\mu(A_i)\| = \Sigma\left\|\int_{A_i} x_i \, d\mu\right\| = \Sigma\|x\|\mu(A_i)$$

and for measurable function x,

$$V_\nu(A) = sup\Sigma\|x\mu(A_i)\| = sup\Sigma\|x_n\mu(A_i)\| = sup\Sigma\|x\|\mu(A_i) = \|x\|\mu(A).$$

Therefore $V_\nu = \|x\|\mu$ and it is finite.

(4) If x is θ, then $\nu(A) = x\mu(A) = \int_A x \, d\mu = \theta = \frac{x}{\|x\|}\|x\|\mu$.

If $x \neq \theta$, then $\nu(A) = x\mu(A) = \frac{x}{\|x\|}\|x\|\mu(A)$.

2-3. LEMMA If x is a Bochner integrable function and $\mu(A) > 0$, then for each $\varepsilon > 0$,

(1) there is a subset B of A such that $\mu(A-B) < \varepsilon$ and $x(B)$ is precompact and

(2) there is a subset C of A with $\mu(C) > 0$ such that $diam\ x(C) < \varepsilon$.

PROOF (1) Since x is Bochner integrable, $\chi_A x$ is the pointwise limit of a sequence $\{x_n\}$ of simple functions and since $\mu(A) < \infty$, this convergence is almost uniform. That is, for $\varepsilon > 0$ there is a subset B of A with $\mu(A\text{-}B) < \varepsilon$ such that $\{x_n\}$ converges uniformly to x on B. But $x_n(B)$ is compact for each n, $x(B)$ is precompact.

(2) By (1), choose $B \subset A$ with $\mu(B) > 0$ and $x(B)$ precompact. Cover $x(B)$ with a finite open sets $\{D_k \subset X: diam(D_k) < \varepsilon, k = 0, 1, \cdots n\}$. For at least one of these, say D_k, $\mu(A \cap x^{-1}(D_k))$ is positive. Set $C = A \cap x^{-1}(D_k)$.

2-4. LEMMA If x is Bochner integrable over A and $\int_A \|x\| d\mu \neq 0$, then

$\int_A x\,d\mu \Big/ \int_A \|x\|\,d\mu$ belongs to the closed convex hull of the set

$$\left\{ x(s) \big/ \|x(s)\| : s\in A,\ x(s)\neq\theta \right\}.$$

PROOF Suppose that x is never θ on A, x vanishes off A and that x is Bochner integrable. Let $\nu=x\mu$, then ν is a vector-valued measure, $V_\nu=\|x\|\mu$ and as we have noted in 2.2 REMARK (4), $\nu=y\cdot V_\nu$, where $y=\frac{x}{\|x\|}$ on A and θ off A. Consequently, by 1.9 THEOREM,

$$\int_A x\,d\mu \Big/ \int_A \|x\|\,d\mu=\frac{\nu(A)}{V_\nu(A)}=\frac{1}{V_\nu(A)}\int_A y\,dV_\nu$$

and it beiongs to the convex hull of $\{y(s): s\in A\}=\left\{\frac{x(s)}{\|x(s)\|} : s\in A\right\}$.

2-5. DEFINITION Let ν be a vector-valued measure on $\mathfrak{A}$ with $V_\nu(S)<\infty$ and $A\in\mathfrak{A}$. The mean direction set of ν on A is defined by

$$\left\{\frac{\nu(B)}{V_\nu(B)} : B\subset A,\ B\in\mathfrak{A},\ V_\nu(B)>0\right\}$$

2-6. COROLLARY If a vector-valued measure ν is a Bochner indefinite integral and $V_\nu(A)>0$, $A\in\mathfrak{A}$ and $\varepsilon<0$, then there is a subset C of A with $V_\nu(C)>0$ such that the diameter of the mean direction set of ν on C is less than ε.

PROOF If $\nu=x\mu$, then by 2.4 LEMMA, the mean direction set of A is a subset of the convex hull of $\frac{x}{\|x\|}(A)$. And by 2.3 LEMMA (2), if $V_\nu(A)>0$ and $\varepsilon>0$, then there is a subset C of A with $V_\nu(C)>0$ such that the diameter of $\frac{x}{\|x\|}(C)$, hence the diameter of its convex hull, is less than ε.

2-7. DEFINITION Let ν be a vector-valued measure on $\mathfrak{A}$.

(1) ν is differentiable iff

(i) $V_\nu(S)<\infty$

(ii) for each $\varepsilon>0$ and $A\in\mathfrak{A}$ with $V_\nu(A)>0$, there is $B\subset A$ with $V_\nu(B)>0$ such that *diam* $\left\{\frac{\nu(C)}{V_\nu(C)} : C\subset B \text{ and } V_\nu(C)>0\right\}<\varepsilon$.

(2) ν is differentiable on $A\in\mathfrak{A}$ iff

(i) $V_\nu(A)<\infty$

(ii) for each $\varepsilon>0$ and $B\subset A$ with $V_\nu(B)>0$, there is $C\subset B$ with $V_\nu(C)>0$ whose mean direction set has diameter less than ε.

2-8. LEMMA (1) Each Bochner indefinite integral is differentiable.

(2) If ν is differentiable on A, then it is differentiable on each subset B of A, $B \in \mathfrak{M}$.

(3) ν is differentiable on set of V_ν-measure 0.

(4) If $V_\nu(S) < \infty$, $A \in \mathfrak{M}$, $A = \bigcup A_n$, where $\{A_n\}$ is a disjoint sequence in $\mathfrak{M}$, and ν is differentiable on each A_n, then ν is differentiable on A.

(5) For a vector-valued measure ν, there exists a set $D \in \mathfrak{M}$ such that ν is differentiable on D and D has maximum V_ν-measure.

PROOF (1) It follows from 2.6 COROLLARY.

(2) Since $V_\nu(B) \leq V_\nu(A) < \infty$ and by 2.6 COROLLARY, ν is differentiable on $B \subset A$ with $B \in \mathfrak{M}$.

(4) $V_\nu(A) = V_\nu(\bigcup A_n) \leq V_\nu(S) < \infty$. For given $\varepsilon > 0$ and $B \subset A$ with $V_\nu(B) > 0$, let $B_n = B \cap A_n$, n=1,2,⋯, then $\{B_n\}$ is a disjoint sequence in $\mathfrak{M}$ whose union is B and since $V_\nu(B) > 0$, there is at least one B_n with $V_\nu(B_n) > 0$. Since ν is differentiable on A_n, for $\varepsilon > 0$ and $B_n \subset A_n$ with $V_\nu(B_n) > 0$, there is $C \subset B_n \subset B$ with $V_\nu(C) > 0$ whose mean direction set has diameter less than ε.

(5) Let $\mathfrak{M}' = \{A \in \mathfrak{M} : \nu \textit{ is differentiable on } A\}$, then $\mathfrak{M}' \neq \phi$. For $A_1, A_2 \in \mathfrak{M}'$, define a order by $A_1 \leq A_2$ if $V_\nu(A_1) \leq V_\nu(A_2)$ and applying Zorn's lemma, there is a maximal element $D \in \mathfrak{M}'$.

2.9 THEOREM If ν is a vector-valued measure with $V_\nu(S) < \infty$, then $\nu = \nu_d + \nu_n$, where ν_d is differentiable, ν_n is nowhere differentiable and ν_d and ν_n are mutually singular.

PROOF By 2.8 LEMMA (5), there is a set $D \in \mathfrak{M}$ on which ν is differentiable and which has maximum V_ν-measure. Put $\nu_d(A) = \nu(A \cap D)$ and $\nu_n(A) = \nu(A - D)$. Then $A = (A \cap D) \cup (A\text{-}D)$, $(A \cap D) \cap (A\text{-}D) = \phi$ and $\nu_d(A\text{-}D) = \nu_n(A \cap D) = 0$, hence ν_d and ν_n are mutually singular. And ν_d is differentiable, ν_n is nowhere differentiable in the sense that it is differentiable on no set of $V_{\nu n}$-positive measure.

2-10. THEOREM A vector-valued measure ν with $V_\nu(S) < \infty$ which is differentiable is a Bochner indefinte integral.

PROOF The set M of vector-valued measures of finite variation is, with total variation as norm, a Banach space, and the map $x \mapsto xV_\nu$ is a norm preserving map of $L_1(V_\nu, X)$ into M. Since $L_1(V_\nu, X)$ is complete, the image is closed, hence it suffices to show that there is a vector-valued function x such that the total variation of xV_ν-ν

is small.

For given $\varepsilon>0$, choose a maximal disjoint family of members A of $\mathfrak{A}$ such that $V_\nu(A)>0$ and *diam* $\{\nu(B)/V_\nu(B): B\subset A$ and $V_\nu(B)>0\}<\varepsilon$. The family of such sets is countable, say $\{A_n:n\in N\}$, because ν has finite total variation, and the complement $S-\bigcup_n A_n$ has V_ν -measure O, since the family is maximal and ν is differentiable. Choose, for each $n\in N$, a member x_n of the mean direction set of A_n and let $x=\Sigma x_n\chi_{A_n}$. Then $\|x\|$, and hence x, is belongs to $L_1(V_\nu, X)$. And, for each $B\in\mathfrak{A}$ and $n\in N$,

$$\int_{B\cap A_n} x\, dV_\nu = x_n V(B\cap A_n)$$

hence

$$\begin{aligned}\| xV_\nu(B\cap A_n)-(B\cap A_n)\| &= \| x_nV_\nu(B\cap A_n)-\nu(B\cap A_n)\| \\ &= \left\| x_n-\frac{\nu(B\cap A_n)}{V_\nu(B\cap A_n)}\right\| V_\nu(B\cap A_n).\end{aligned}$$

Since x_n belongs to the mean direction set of A_n whose diameter is less than ε,

$$\| xV_\nu(B\cap A_n)-\nu(B\cap A_n)\| <\varepsilon V_\nu(B\cap A_n).$$

The sequences $\{xV_\nu(B\cap A_n):n\in N\}$ and $\{\nu(B\cap A_n):n\in N\}$ are absolutely summable and consequently

$$\begin{aligned}\| xV_\nu(B)-\nu(B)\| &= \| \sum_n xV_\nu(B\cap A_n)-\sum_n\nu(B\cap A_n\| \\ &\leq \sum_n \| xV_\nu(B\cap A_n)-\nu(B\cap A_n)\| \\ &< \varepsilon\cdot V_\nu(B).\end{aligned}$$

Thus $\| xV_\nu(B)-\nu(B)\| <\varepsilon V_\nu(B)$ for all $B\in\mathfrak{A}$, therefore $\| xV-\nu\| \leq\varepsilon V\nu(S)$.

REFERENCES

1. J.L. Kelley and T.P. Srinivasan, *On the Bochner Integral, Vector and Operator Valued Measures and Applications*, Academic Press Inc., New York, U.S.A. (1973), pp.165~174.
2. B.J. Pettis, *On Integration in Vector Spaces*, Trans. Amer. Soc. Vol.44 (1938), pp.277~304.
3. K.Yosida, *Functional Analysis*, Kinokuniya Book-Store Co., Ltd. Tokyo Japan, (1971).

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