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AN ANALYTIC SUFFICIENCY CONDITION FOR GOLDBACH'S CONJECTURE WITH MINIMAL REDUNDANCY

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1. Introduction

In this paper we present an analytic sufficiency condition for Goldbach's conjecture which eliminates approximately seventy-five percent of the redundancy inherent in all other known Hardy-Littlewood-Vinogradov circle method, analytic sufficiency conditions.

2. Notations and definitions

Let p denote a prime. Let $n \geq 2$ denote an integer. Let $x_0 = \left(\frac{\log^{15} n}{n}\right)$.
Let

$$f_c(x,n) = \sum_{p \leq n} \cos\ (2\pi p x),$$

$$f_s(x,n) = \sum_{p \leq n} \sin\ (2\pi p x),$$

$$f(x,n) = f_c(x,n) + i f_s(x,n).$$

Let

$$g_c(x,v) \sum_{2 \leq m \leq v} \frac{\cos(2\pi m x)}{\log m};\ g_c(x,v) = 0 \text{ if } v < 2.$$

Let

$$g_s(x,v) = \sum_{2 \leq m \leq v} \frac{\sin(2\pi m x)}{\log m};\ g_s(x,v) = 0 \text{ if } v < 2.$$

Let $g(x,v) = g_c(x,v) + i g_s(x,v)$. For each n let $m(n)$ be those points in $[x_0, x_0+1]$ which are not in any closed neighborhood (major arc) of radius x_0 about any rational number $\frac{h}{q}$ where $(h,q)=1$, $(q,n)=1$, and $q \leq \log^{15} n$. Let $M(n) = [x_0, x_0+1] - m(n)$. Let $r(n)$ be the number of representations of n as the sum of two primes.

3. Main result

By a straightforward computation it can be shown that

$$r(n)=2\int_{x_0}^{x_0+1}[f_c^2(x,n)-f_s^2(x,n)]\cos(2\pi nx)dx \text{ for any } x_0$$

$$=2\int_{m(n)}[f_c^2(x,n)-f_s^2(x,n)]\cos(2\pi nx)dx$$

$$+2\int_{M(n)}[f_c^2(x,n)-f_s^2(x,n)]\cos(2\pi nx)dx$$

$$=A(n)+B(n).$$

THEOREM 1. *$A(n)=0\ (n\log^{-2}n)$ implies $r(n)>0$ for every even $n\geq N_0$.*

4. Some comments

$$|A(n)|\leq 2\int_0^1 f_c^2(x,n)dx+2\int_0^1 f_s^2(x,n)dx=\pi(n)+\pi(n);$$

so that

$$A(n)=0(n\log^{-1}n).$$

It is known (cf. [7] or [8] for details) that if $A^*(n)$ is defined:

$$\int_{m(n)}[f_c(x,n)+if_s(x,n)]^2[\cos(2\pi px)+i\sin(2\pi px)]dx,$$

then $A^*(n)=0\ (n\log^{-2}n)$ implies $r(n)>0$ for every even $n\geq N_0$.

Let $R(x,n)$ and $I(x,n)$ be the real and imaginary parts, respectively, of the integrand of $A^*(n)$. It is easy to see that

$$R(x,n)=[f_c^2(x,n)-f_s^2(x,n)]\cos(2\pi nx)+2f_c(x,n)f_s(x,n)\sin(2\pi nx),$$

and

$$I(x,n)=-[f_c^2(x,n)-f_s^2(x,n)]\sin(2\pi nx)+2f_c(x,n)f_s(x,n)\cos(2\pi nx).$$

All previously known Hardy-Littlewood-Vinogradov circle method, analytic sufficiency conditions for Goldbach's conjecture are similar to that given above involving $A^*(n)$. The basic differences between them arise from the manner in which $m(n)$ is defined, and whether or not one wishes to invoke the generalized Riemann hypothesis (cf. [7] or [8]).

To remove the $(q,n)=1$ condition in the definition of $m(n)$ in theorem 1 one can let $q\leq n^{\varepsilon}$ and then invoke the generalized Riemann hypothesis (cf. [7] or [8]). It might be possible to do this by letting $q\leq n^{\varepsilon}$ and then using the techniques in [3] to handle the exceptional character; so that the generalized Riemann hypothesis can be avoided.

Theorem 1 was conjectured while the author was trying to simplify a computer program employed to analyze the pointwise behavior of $R(x,n)$ and $I(x,n)$. The results of this analysis as well as the analysis of the pointwise behavior of the integrand in the hypothesis of theorem 1 is presented in [5].

5. Proof of theorem 1

Fix $n \geq N_0$. By definition

$$2\int_{m(n)} [f_c^2(x,n) - f_s^2(x,n)] \cos(2\pi nx)dx = 2 \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \sum_{\substack{0<h\leq q \\ (h,q)=1}} T(h,q)$$

where

$$T(h,q) = T_c(h,q) - T_s(h,q),$$

$$T_c(h,q) = \int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} f_c^2(x,n) \cos(2\pi nx)dx,$$

and

$$T_s(h,q) = \int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} f_s^2(x,n) \cos(2\pi nx)dx.$$

LEMMA 1. *Let* $q \leq \log^{15} n$, $|y| \leq x_0$, $(h,q)=1$ *and* $n \geq N_0$. *Then*

$$\left| f_c\left(\frac{h}{q}+y,n\right) - \frac{\mu(q)}{\phi(q)} g_c(y,n) \right| \leq n \log^{-69} n$$

and

$$\left| f_s\left(\frac{h}{q}+y,n\right) - \frac{\mu(q)}{\phi(q)} g_s(y,n) \right| \leq n \log^{-69} n.$$

PROOF. This follows from theorem 58 in [1].

LEMMA 2. *Under the hypothesis of lemma* 1 *we have*

$$\left| f_c^2\left(\frac{h}{q}+y,n\right) - \frac{\mu^2(q)}{\phi^2(q)} g_c^2(y,n) \right| \leq 2n^2 \log^{-69} n$$

and

$$\left| f_s^2\left(\frac{h}{q}+y,n\right) - \frac{\mu^2(q)}{\phi^2(q)} g_s^2(y,n) \right| \leq 2n^2 \log^{-69} n.$$

PROOF. This is immediate by lemma 1 and the trivial inqualities $|f_c(x,n)| \leq n$, $|f_s(x,n)| \leq n$, $|g_c(y,n)| \leq n$ and $|g_s(y,n)| \leq n$ and the fact that if $|a| \leq n$ and $|b| \leq n$, then $|a^2-b^2| \leq 2n|a-b|$.

By a change of variable $y=\left(x-\frac{h}{q}\right)$ we have

$$T_c(h,q)=\int_{-x_0}^{x_0} f_c^2\left(\frac{h}{q}+y\right)\cos\left(2\pi n\left(y+\frac{h}{q}\right)\right)dy$$

$$=\cos\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} f_c^2\left(\frac{h}{q}+y\right)\cos(2\pi ny)\,dy$$

$$-\sin\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} f_c^2\left(\frac{h}{q}+y\right)\sin(2\pi ny)\,dy, \qquad \text{(A)}$$

and

$$T_s(h,q)=\int_{-x_0}^{x_0} f_s^2\left(\frac{h}{q}+y\right)\cos\left(2\pi n\left(y+\frac{h}{q}\right)\right)dy$$

$$=\cos\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} f_s^2\left(\frac{h}{q}+y\right)\cos(2\pi ny)\,dy$$

$$-\sin\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} f_s^2\left(\frac{h}{q}+y\right)\sin(2\pi ny)\,dy. \qquad \text{(B)}$$

However, by lemma 2 and (A) we have under the hypothesis of lemma 2

$$\left|\left(\cos\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} f_c^2\left(\frac{h}{q}+y\right)\cos(2\pi ny)\,dy-\frac{\mu^2(q)}{\phi^2(q)}\cos\left(2\pi n\frac{h}{q}\right)\right.\right.$$

$$\left.\times\int_{-x_0}^{x_0} g_c^2(y,n)\cos(2\pi ny)\,dy\right)-\left(\sin\left(2\pi n\frac{h}{q}\right)\right.$$

$$\times\int_{-x_0}^{x_0} f_c^2\left(\frac{h}{q}+y\right)\sin(2\pi ny)\,dy-\frac{\mu^2(q)}{\phi^2(q)}\sin\left(2\pi n\frac{h}{q}\right)$$

$$\left.\left.\times\int_{-x_0}^{x_0} g_c^2(y,n)\sin(2\pi ny)\,dy\right)\right|$$

$$\leq 2\int_{-x_0}^{x_0}\left|f_c^2\left(\frac{h}{q}+y\right)-\frac{\mu^2(q)}{\phi^2(q)}g_c^2(y,n)\right|dy$$

$$\leq 4\int_{-x_0}^{x_0} n^2\log^{-69}n\ dy$$

$$=8x_0n^2\log^{-69}n=8n\log^{-54}n.$$

So that if $(h,q)=1$ and $q\leq\log^{15}n$, then

$$\left|T_c(h,q)-\frac{\mu^2(q)}{\phi^2(q)}\left[\cos\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} g_c^2(y,n)\cos(2\pi ny)dy\right]\right.$$

$$\left.+\frac{\mu^2(q)}{\phi^2(q)}\left[\sin\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} g_c^2(y,n)\sin(2\pi ny)dy\right]\right|$$

$$\leq 8n\log^{-54}n. \tag{C}$$

By a similar argument we have that if $(h,q)=1$ and $q\leq\log^{15}n$, then

$$\left|T_s(h,q)-\frac{\mu^2(q)}{\phi^2(q)}\left[\cos\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} g_s^2(y,n)\cos(2\pi ny)dy\right]\right.$$

$$\left.+\frac{\mu^2(q)}{\phi^2(q)}\left[\sin\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0} g_s^2(y,n)\sin(2\pi ny)dy\right]\right|$$

$$\leq 8n\log^{-54}n \tag{D}$$

By (C) and (D) we have that if $(h,q)=1$ and $q\leq\log^{15}n$, then

$$\left|T(h,q)-\frac{\mu^2(q)}{\phi^2(q)}\left[\cos\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0}\left[g_c^2(y,n)-g_s^2(y,n)\right]\cos(2\pi ny)dy\right]\right.$$

$$\left.+\frac{\mu^2(q)}{\phi^2(q)}\left[\sin\left(2\pi n\frac{h}{q}\right)\int_{-x_0}^{x_0}\left[g_c^2(y,n)-g_s^2(y,n)\right]\sin(2\pi ny)dy\right]\right|$$

$$\leq 16n\log^{-54}n. \tag{E}$$

Let

$$T_c(n)=\int_{-x_0}^{x_0}[g_c^2(y,n)-g_s^2(y,n)]\cos(2\pi ny)dy.$$

Let

$$T_s(n)=\int_{-x_0}^{x_0}[g_c^2(y,n)-g_s^2(y,n)]\sin(2\pi ny)dy.$$

Let

$$T(n)=\frac{1}{2}\sum_{m_1,m_2}(\log^{-1}m,\ \log^{-1}m_2) \tag{F}$$

with the conditions of summation $m_1\geq 2$, $m_2\geq 2$, and $m_1+m_2=n$.

It is easy to see by a straightforward calculation that

$$T(n)=\int_{-1/2}^{1/2}[g_c^2(y,n)-g_s^2(y,n)]\cos(2\pi ny)dy \tag{G}$$

Also, it is clear that the number of terms on the right-hand side of (F) is $(n-3)$, and each term is greater than $\log^{-2}n$ and less than 1; so that

$$\frac{1}{6}n\log^{-2}n<T(n). \tag{H}$$

It is easy to see using the formula for the sum of a geometric series that

$$\left|\sum_{m=2}^{m_1}\varepsilon(my)\right|\leq\frac{1}{|\sin(\pi y)|}\leq\frac{1}{2|y|};\ \left(m_1\geq 2,\ 0<|y|\leq\frac{1}{2}\right).$$

Hence by the definition of $g(y,n)$ and Abel's lemma,

$$|g(y,n)|\leq|y|^{-1}\ \left(0<|y|\leq\frac{1}{2}\right);$$

so that

$$|g_c^2(y,n)-g_s^2(y,n)|\leq 2|y|^{-2};$$

so that

$$|T(n)-T_c(n)|\leq 4\int_{x_0}^{1/2}y^{-2}dy\leq 4x_0^{-1}=4n\log^{-15}n;$$

so that for $(h,q)=1$ and $q\leq\log^{15}n$

$$\left|\cos\left(2\pi n\frac{h}{q}\right)\right|\left|\frac{\mu^2(q)}{\phi^2(q)}\right||T(n)-T_c(n)|\leq\frac{1}{\phi^2(q)}(4n\log^{-15}n) \tag{I}$$

By (E) and (I) we have that if $(h,q)=1$ and $q\leq\log^{15}n$, then

$$\left|T(h,q)-\frac{\mu^2(q)}{\phi^2(q)}\cos\left(2\pi n\frac{h}{q}\right)T(n)+\frac{\mu^2(q)}{\phi^2(q)}\sin\left(2\pi n\frac{h}{q}\right)T_s(n)\right|$$

$$\leq 16n\log^{-54}n+\frac{1}{\phi^2(q)}(4n\log^{-15}n); \tag{J}$$

so that adding (J) $\phi(q)$ times for some fixed $q\leq\log^{15}n$ we have

$$\Big|\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h,q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)\sum_{\substack{0<h\leq q\\(h,q)=1}}\cos\Big(2\pi n\frac{h}{q}\Big)$$
$$+\frac{\mu^2(q)}{\phi^2(q)}T_s(n)\sum_{\substack{0<h\leq q\\(h,q)=1}}\sin\Big(2\pi n\frac{h}{q}\Big)\Big|$$
$$\leq(16n\log^{-54}n)\,\phi(q)+\frac{1}{\phi^{4/3}(q)}(4n\log^{-15}n)\,\phi^{1/3}(q). \qquad \text{(K)}$$

But $\phi(q)\leq\log^{15}n$ and by definition

$$C_q(n)=\sum_{\substack{0<h\leq q\\(h,q)=1}}\cos\Big(2\pi n\frac{h}{q}\Big)-i\sum_{\substack{0<h\leq q\\(h,q)=1}}\sin\Big(2\pi n\frac{h}{q}\Big),$$

and the imaginary part of $C_q(n)$ is equal to zero; so that

$$\Big|\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h,q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)C_q(n)\Big|$$
$$\leq 16n\log^{-39}n+\frac{1}{\phi^{4/3}(q)}(4n\log^{-10}n). \qquad \text{(L)}$$

Now summing over all $q\leq\log^{15}n$ such that $(q,n)=1$ we have

$$\Big|2\sum_{\substack{q\leq\log^{15}n\\(q,n)=1}}\ \sum_{\substack{0<h\leq q\\(h,q)=1}} T(h,q)-2T(n)\sum_{\substack{q\leq\log^{15}n\\(q,n)=1}}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\Big|$$
$$\leq(32n\log^{-39}n)(\log^{15}n)+\Big[\sum_{q\leq\log^{15}n}\frac{1}{\phi^{4/3}(q)}\Big](8n\log^{-10}n)$$
$$\leq 32n\log^{-24}n+C_1(8n\log^{-10}n)\leq C_2 n\log^{-10}n; \qquad \text{(M)}$$

since by theorem 327 in [2]

$$\sum_{q\leq\log^{15}n}\frac{1}{\phi^{4/3}(q)}\leq C_1 \ (C_1 \text{ independent of } n).$$

Hence by (M) and the hypothesis we have

$$\Big|r(n)-2T(n)\sum_{\substack{q\leq\log^{15}n\\(q,n)=1}}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\Big|\leq k(n)n\log^{-2}n+C_2 n\log^{-10}n \qquad \text{(N)}$$

where $k(n)\to 0$.

Now let

$$R(n)=\sum_{\substack{q\leq\log^{15}n\\(q,n)=1}}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)$$

and

$$S(n)=\sum_{q=1}^{\infty}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\,D_q(n)$$

where

$$D_q(n)=\begin{cases}1 \text{ if } (q,n)=1\\ 0 \text{ if } (q,n)>1\end{cases}$$

Then

$$|R(n)-S(n)|=\Big|\sum_{q>\log^{16} n}\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\,D_q(n)\Big|$$

$$\leq \sum_{\substack{q>\log^{16} n\\ q \text{ square free}}}\frac{1}{\phi^2(q)};$$

since $\mu^2(q)=0$ if q is not square free, and by theorem 272 in [2] if q is square free and $(q,n)=1$, then $|C_q(n)|=1$. Hence $|S(n)-R(n)|\leq C_3\log^{-14}n$, by theorem 327 in [2]; so that $S(n)-R(n)=0(1)$.

By (H) what remains to be done is to show that $S(n)$ is uniformly bounded away from zero.

Let

$$f(q)=\frac{\mu^2(q)}{\phi^2(q)}C_q(n)\,D_q(n).$$

Since $\mu(q)$, $\phi(q)$, $D_q(n)$ and $C_q(n)$ are all multiplicative fuctions of q, f is a multiplicative function of q. Also, by means of the trivial estimate on $C_q(n)$, namely n, and a direct application of theorem 327 in [2] we have

$$\sum_{q=1}^{\infty}|f(q)|\leq n\sum_{q=1}^{\infty}\frac{1}{\phi^2(q)}<\infty \quad \text{for each } n;$$

so that by theorem 2 in [1] we have for each n

$$S(n)=\prod_{p}\sum_{m=0}^{\infty}f(p^m).$$

But

If $m=0$, $f(p^m)=f(p^0)=f(1)=\dfrac{\mu^2(1)}{\phi^2(1)}C_1(n)\,D_1(n)=1$.

If $m=1$, $f(p^1)=f(p)=\dfrac{\mu^2(p)}{\phi^2(p)}C_p(n)\,D_p(n)=\dfrac{C_p(n)\;D_p(n)}{(p-1)^2}$.

If $m\geq 2$, $\mu(p^m)=0$; so that $f(p^m)=0$; so that

$$S(n)=\prod_{p}\left(1+\frac{C_p(n)D_p(n)}{(p-1)^2}\right).$$

Clearly, if n is even, $D_2(n)=0$; and by theorem 272 in [2] we have $C_p(n)=(p-1)$ if $(p,n)>1$ and $C_p(n)=-1$ if $(p,n)=1$; so that

$$S(n)=\prod_{p>2}\left(1+\frac{C_p(n)D_p(n)}{(p-1)^2}\right)\geq\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$
$$\geq\prod_{m=2}^{\infty}\left(1-\frac{1}{m^2}\right)=\frac{1}{2};$$

so that theorem 1 is now established.

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