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ČECH CLOSURE SPACES

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1. Introduction

Nearness spaces, introduced by Herrlich [9], provides one of the most unifying concepts to appear in topology in recent years. The categories of symmetric topological spaces, uniform spaces, proximity spaces and contiguity spaces are all embedded in the category NEAR, of nearness spaces and nearness maps.

Nearness spaces have had an impact on the study of extensions of a topological space; for example [1], [2] and [6].

The underlying structure of each nearness space is a topological space. A slight modification of one axiom of a nearness space yields a semi-nearness space, called a Čech nearness space in [11]. The underlying structure for each semi-nearness space is a Čech closure space. It is this fact that has provided the motivation for this paper. Čech closure spaces were introduced by Čech [7]. Thron, in [13], has studied the lattice of semi-proximites compatible with a given Čech closure space.

Herrlich, in [10], notes several properties that fail to hold for topological spaces but which do hold for semi-nearness spaces. In this paper, for example, it is noted that the composition of two topological closure operators on a given set need not be a topological closure operator but it is a Čech closure operator.

Sharp [12] and Bonnett and Porter [5] study finite topological spaces and they represent them using zero-one matrices. Many topological properties are characterized in terms of the corresponding matrices in these two papers.

In this paper it is shown that a Čech closure operator on a finite set may be represented by a zero-one reflexive matrix. A number of separation properties are studied and for finite spaces characterized in terms of the matrix that represents the closure operator. It is also shown that Čech closure spaces satisfying certain mild separation axioms are topological spaces.

For each Čech closure space there exists an underlying topology, defined in a natural way. Separation properties that carry over to the underlying topology are studied. Also, a Čech closure operator generates a semi-topology; that is, a "topology" without the union axiom. This is closely related to the work by

Bentley and Slepian [4], although there they also modify the intersection axiom.

Finitely generated Čech closure spaces are a natural generalization of finite Čech closure spaces. It is shown that the collection of all finitely generated Čech closure spaces on a given set, partially ordered in a natural way, yields a uniquely complemented distributive complete lattice and hence a Boolean algebra.

Čech closure spaces of finite degree provide a non-trivial generalization of topological spaces. It is shown that the category of topological spaces and continuous maps is bi-reflective in the category of Čech closure spaces of finite degree and continuous maps.

2. Preliminaries

DEFINITION 2.1. Let X be a non-empty set and $c:P(X)\to P(X)$ satisfying:

(1) $c(\phi)=\phi$;

(2) $A\subset c(A)$ for each $A\subset X$;

(3) $c(A\cup B)=c(A)\cup c(B)$ for all subsets A and B of X.

Then (X,c) is called a *Čech closure space.* (X,c) is called a *topological space,* and c a *Kuratowski* or *topological closure operator* if c also satisfies:

(4) $c(c(A))=c(A)$ for each $A\subset X$.

Let (X,c) be a Čech closure space. Set $t(c)=\{O\subset X : c(X-O)=X-O\}$. Easily $t(c)$ is a topology on X, and it is called the *underlying topology of* (X,c). If $A\subset X$ then $\overline{A}$ denotes the closure of A with respect to the underlying topology $t(c)$.

DEFINITION 2.2. Let (X,c) and (Y,d) be Čech closure spaces. Let $f: X\to Y$. f is said to be *continuous* provided $f(c(A))\subset d(f(A))$ for each $A\subset X$.

The following result is found in Čech [9].

LEMMA A. *Let (X,c) be a Čech closure space. Let $A\subset B\subset X$. Then $c(A)\subset c(B)$.*

DEFINITION 2.3. Let (X,c) be a Čech closure space. (X,c) is called *finitely generated* provided $c(A)=\bigcup\{c(a) : a\in A\}$.

Easily each finite Čech closure space is finitely generated.

THEOREM 2.1. *Let X be a set and $e: X\to P(X)$ satisfying:*

$$(1) \qquad \{x\}\subset e(x) \text{ for each } x\in X.$$

Let $c: P(X)\to P(X)$ be defined by

$$c(A)=\begin{cases}\bigcup\{e(a): a\in A\} & \text{if } A\neq\phi,\\ \phi & \text{if } A=\phi.\end{cases}$$

Then (X, c) is a finitely generated Čech closure operator and; moreover, each finitely generated Čech closure space can be constructed in this manner.

The essential point is that a finitely generated Čech closure operator is completely determined by its action on singleton sets.

A topology on a finite set can be represented by a zero-one matrix as demonstrated by Sharp [12]. A similar representation is possible for a finite Čech closure space. Let (S, c) be a finite Čech closure space. Denote S by: $S=\{s_1, s_2, \cdots s_n\}$. Define a matrix $T_c={}_n[t_{ij}]_n$ by:

$$t_{ij}=\begin{cases}1 & \text{if } j\in c(i),\\ 0 & \text{if } j\notin c(i).\end{cases}$$

A basic result, found in Sharp[12], is that an $n\times n$ zero-one matrix T represents a topology on S if and only if it is reflexive and transitive; that is, $t_{ii}=1$ for each i, and $T^2=T$, where the matrix multiplication is with respect to Boolean arithmetic. In a similar manner it is evident that a $n\times n$ zero-one matrix T represents a Čech closure operator on S if and only if it is reflexive.

The following notation will be used for zero-one matrices P and Q of order $n\times n$:

$$P\vee Q={}_n[p_{ij}\vee q_{ij}]_n$$
$$P\wedge Q={}_n[p_{ij}\wedge q_{ij}]_n.$$

If P and Q are vectors then $P\vee Q$ and $P\wedge Q$ are defined similarly. The matrix product of the matrices P and Q, with respect to Boolean arithmetic, can be expressed as $[\bigvee_{k=1}^{n}(p_{ik}\wedge q_{kj})]$. P_i denotes the i-th row of the matrix P. The identity matrix is denoted by I.

Let (S, c) be a finite Čech closure space and let T_c be the matrix representing c. The following notation corresponds to that found in Bonnett and Porter [5], and Sharp [12]. For each $s_i\in S$, let $\varepsilon_i=(\delta_{1i}, \delta_{2i}, \cdots, \delta_{ni})$, where δ_{ij} is Kronecker's delta. For $A\subset S$, let A_v be the vector defined by $A_v=\vee\{\varepsilon_i : s_i\in A\}$. Easily, $T_i=(c\{s_i\})_v$.

THEOREM 2.2. *Let (S, c) be a finite Čech closure space and let T_c denote the matrix representing c. If $A\subset S$, then $(c(A))_v=(A_v)T_c$, where the multiplication is with respect to Boolean arithmetic.*

PROOF. The proof is similar to the proof for the corresponding result for finite topological spaces found in Bonnett and Porter [5].

DEFINITION 2.4. Let (S, c) be a finite Čech closure operator and let T_c denote the matrix representing c. c^T denotes the Čech closure operator on S corresponding to the reflexive matrix T_c^T, the transpose of T_c.

3. Basic results

DEFINITION 3.1. Two Čech closure spaces (X, c) and (Y, d) are called *closure homeomorphic* provided there exists a one-to-one and onto map $f: X \to Y$ such that $f(c(A)) = d(f(A))$ for each $A \subset X$.

DEFINITION 3.2. Let (X, c) and (X, d) be Čech closure spaces. Then, for each $A \subset X$:

(a) $(c \cup d)(A) = c(A) \cup d(A)$

(b) $(c \cap d)(A) = c(A) \cap d(A)$

(c) $c < d$ if and only if $c(A) \subset d(A)$ for each $A \subset X$.

Let $L(X)$ denote the collection of all Čech closure operators on X and $C(X)$ the collection of all finitely generated Čech closure operators on X. Then $(L(X), <)$ and $(C(X), <)$ are partially ordered sets.

THEOREM 3.1. *Let T and T^* be the two matrices corresponding to the finite Čech closure spaces (S, c) and (S, c^*), respectively. The following statements are equivalent.*

(1) *(S, c) and (S, c^*) are closure homeomorphic.*

(2) *There exists a permutation matrix P such that $T^* = P^T T P$.*

PROOF. Let $S = \{s_1, \cdots, s_n\}$ be a finite set and c and c^* Čech closure operators on S, represented, respectively, by the matrices $T = [t_{ij}]$ and $T^* = [t^*_{ij}]$. Then (S, c) is closure homeomorphic to (S, c^*) if and only if there exists a permutation $\pi: S \to S$ such that $\pi(c(s_i)) = c^*(\pi(s_i))$ for $1 \leq i \leq n$.

Let $\pi: S \to S$ be any permutation, represented by the matrix $P = [p_{ij}]$ defined by

$$p_{ij} = \begin{cases} 1 & \text{if } s_j = \pi(s_i), \\ 0 & \text{otherwise.} \end{cases}$$

We first show:

(A) $(TP)_i = (\pi(c(s_i)))_v$ for $1 \leq i \leq n$, and

(B) $(PT^*)_i = (c^*(\pi(s_i)))_v$ for $1 \leq i \leq n$.

Proof of (A). Let $1\leq j\leq n$ and let $U=[u_{ij}]=TP=[\sum_{k=1}^{n} t_{ik}p_{kj}]$. Now there exists k' such that $s_j=\pi(s_{k'})$. Then $u_{ij}=t_{ik'}$, and;

$t_{ik'}=1 \Longleftrightarrow s_{k'}\in c(s_i) \Longleftrightarrow s_j\in\pi(c(s_i))$, and

$t_{ik'}=0 \Longleftrightarrow s_{k'}\notin c(s_i) \Longleftrightarrow s_j\notin\pi(c(s_i))$.

Hence $(TP)_i=(\pi(c(s_i)))_v$ for $1\leq i\leq n$.

Proof of (B). Let $1\leq j\leq n$ and let $U=[u_{ij}]=PT^*=[\sum_{k=1}^{n} p_{ik}t^*_{kj}]$. Again, there exists k' such that $s_{k'}=\pi(s_i)$. Thus $u_{ij}=p_{ik'}t^*_{k'j}=t^*_{k'j}$. Now;

$t^*_{k'j}=1 \Longleftrightarrow s_j\in c^*(s_{k'}) \Longleftrightarrow s_j\in c^*(\pi(s_i))$; and

$t^*_{k'j}=0 \Longleftrightarrow s_j\notin c^*(s_{k'}) \Longleftrightarrow s_j\notin c^*(\pi(s_i))$.

Thus, $(PT^*)_i=(c^*(\pi(s_i)))_v$ for $1\leq i\leq n$.

The proof of the theorem is completed by noting that the following statements are equivalent.

(1) $\pi:(S,c)\rightarrow(S,c^*)$ is a closure homeomorphism.

(2) $\pi(c(s_i))=c^*(\pi(s_i))$ for $1\leq i\leq n$.

(3) $(PT^*)_i=(TP)_i$ for $1\leq i\leq n$.

(4) $PT^*=TP$

(5) $T^*=P^TTP$

The proof of the following theorem is straight-forward and omitted.

THEOREM 3.2. *Let X be a non-empty set and c_α a Čech closure operator on X for each $\alpha\in\Lambda$. Define d by $d(A)=\bigcup\{c_\alpha(A):\alpha\in\Lambda\}$. Then*:

(1) *d is a Čech closure operator on X.*

(2) *$d=\bigvee\{c_\alpha:\alpha\in\Lambda\}$.*

(3) *$(L(X),\vee,\wedge)$ is a complete lattice.*

COROLLARY 3.3. *Let (X,c) and (X,d) be Čech closure spaces. Then $c\cup d$ is a Čech closure operator on X, and $c\cup d=c\vee d$.*

COROLLARY 3.4. *$(C(X),\vee,\wedge)$ is a complete lattice.*

PROOF. The proof follows easily from theorem 3.3 and the fact that $\bigcup\{c_\alpha:\alpha\in\Lambda, c_\alpha\in C(X)\}$ is finitely generated.

EXAMPLE 3.1. Let c and d be Čech closure operators on X; then $c\cap d$ need not be a Čech closure operator on X. Let $X=\{1,2,3\}$. Let c and d be defined by:

$c(1)=\{1,2\}$ $\qquad$ $d(1)=\{1\}$

$c(2)=\{2\}$ $\qquad$ $d(2)=\{2\}$

$c(3)=\{3\}$ $\qquad$ $d(3)=\{2,3\}$.

Let $A=\{1\}$ and $B=\{3\}$. Then: $(c\cap d)(A)\cup(c\cap d)(B)=\{1,3\}$, but $(c\cap d)(A\cup B)=\{1,2,3\}$. Hence $c\cap d$ is not a Čech closure operator.

As example 3.1 shows $c\cap d$ need not be a Čech closure operator. As the following two theorems demonstrate, however, if c and d are finitely generated, or X is finite, then one can determine $c\wedge d$ easily.

THEOREM 3.5. *Let (X,c) and (X,d) be finitely generated Čech closure spaces. For each $x\in X$, let $e(x)=c(x)\cap d(x)$. Then $(c\wedge d)(A)=\cup\{e(x): x\in A\}$, where $c\wedge d$ is with respect to $C(X)$.*

PROOF. Let $e(A)=\cup\{e(x): x\in A\}$ for each $A\subset X$. Then by theorem 2.1, $e\in C(X)$. Let $A\subset X$. Then $(c\wedge d)(A)\subset c(A)$ and $(c\wedge d)(A)\subset d(A)$. Easily $e(A)\subset c(A)$ and $e(A)\subset d(A)$. Hence $c\wedge d>e$. Now $(c\wedge d)(x)\subset c(x)\cap d(x)\subset e(x)$ for each $x\in X$. Hence $c\wedge d<e$ since $c\wedge d$ is finitely generated.

THEOREM 3.6. *Let (S,c) and (S,d) be finite Čech closure spaces represented by the matrices T_c and T_d, respectively. Then:*

(1) $T_{c\vee d}=T_c\vee T_d$, *and*

(2) $T_{c\wedge d}=T_c\wedge T_d$.

PROOF. Statement (1) follows from corollary 3.3 and statement (2) follows from theorem 3.5.

THEOREM 3.7. *$(C(X),\vee,\wedge)$ is a distributive lattice.*

PROOF. Let a, b, and c belong to $C(X)$, and $A\subset X$. Then:

$$\begin{aligned}(a\vee(b\wedge c))(A)&=\cup\{(a(x)\cup(b(x)\cap c(x)): x\in A\}\\&=\cup\{(a(x)\cup b(x))\cap(a(x)\cup c(x)): x\in A\}\\&=((a\vee b)\wedge(a\vee c))(A).\end{aligned}$$

In a similar manner, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$.

THEOREM 3.8. *Let $c\in C(X)$. For each $x\in X$, let $e(x)=\{x\}\cup(X-c(x))$. Define c' by $c'(A)=\cup\{e(x): x\in X\}$. Then:*

(1) $c'\in C(X)$.

(2) *c' is a complement of c in $(C(X),\vee,\wedge)$.*

(3) *$(C(X),\vee,\wedge)$ is a uniquely complemented lattice.*

PROOF. (1). Let A and B be subsets of X. Then:

$$c'(A\cup B)=\cup\{e(x):x\in A\cup B\}$$
$$=(\cup\{e(x):x\in A\})\cup(\cup\{e(x):x\in B\})$$
$$=c'(A)\cup c'(B).$$

It easily follows that $c'\in C(X)$.

(2). $(c\vee c')(A)=(c\cup c')(A)=c(A)\cup(\cup\{e(x):x\in A\})$
$$=c(A)\cup(\cup\{\{x\}\cup(X-c(x)):x\in A\})$$
$$=X.$$

Now by theorem 3.5 ; $(c\wedge c')(A)=\cup\{c(a)\cap c'(a):a\in A\}$
$$=\cup\{c(a)\cap(\{a\}\cup(X-c(a))):a\in A\}$$
$$=A.$$

(3). Since $(C(X),\vee,\wedge)$ is a distributive lattice by theorem 3.7 and has complements by (2), it follows that it is uniquely complemented.

COROLLARY 3.9. *$(C(X),\vee,\wedge)$ is a Boolean algebra.*

THEOREM 3.10. *Let (S,c) be a finite Čech closure space. Let T_c denote the matrix representing c. Then $T_{c'}=[r_{ij}]$ where:*

$$r_{ij}=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j \text{ and } t_{ij}=1\\ 1 \text{ if } i\neq j \text{ and } t_{ij}=0.\end{cases}$$

COROLLARY 3.11. *Let (S,c) be a finite Čech closure space. Then $(c')^T=(c^T)'$.*

EXAMPLE 3.2. Two Čech closure operators may be complements and their underlying topologies not be complements. Let $S=\{s_1,s_2,s_3\}$. Let c be the Čech closure operator on S represented by the matrix:

$$T_c=\begin{bmatrix}1&1&0\\0&1&0\\1&0&1\end{bmatrix}. \qquad \text{Then; } T_{c'}=\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}.$$

Now $t(c)=t(c')=\{S,\phi\}$, the trivial topology on S.

THEOREM 3.12. *Let (X,c) and (X,d) be Čech closure spaces. Then:*
$$t(c\vee d)=t(c)\wedge t(d).$$

PROOF. By corollary 3.3, $t(c\cup d)=t(c\vee d)$. Let $O\in t(c\cup d)$. Then $(c\cup d)(X-O)=X-O$ and $c(X-O)\cup d(X-O)=X-O$. Hence $c(X-O)=X-O$ and $d(X-O)=X-O$. Therefore, $O\in t(c)\cap t(d)=t(c)\wedge t(d)$.

Let $O\in t(c)\cap t(d)$. Then $c(X-O)=X-O$ and $d(X-O)=X-O$ and thus $(c\cup d)(X-O)=X-O$, and $O\in t(c\cup d)$. Therefore, $t(c\vee d)=t(c)\wedge t(d)$.

Let c and d be Čech closure operators on a set X. The following example shows that $t(c\wedge d)$ need not equal $t(c)\vee t(d)$.

EXAMPLE 3.3. Let $X=\{1,2,3\}$ and c and d be Čech closure operators on X represented by:

$$T_c=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix} \qquad \text{and } T_d=\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}.$$

Then $t(c)$ and $t(d)$ are the trivial topology on X and $t(c\wedge d)$ is the discrete topology on X, since by theorem 3.6,

$$T_{c\wedge d}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

4. Composition of Čech closure operators

DEFINITION 4.1. Let c and d be Čech closure operators on a set X. Then $c\circ d$ is defined by $(c\circ d)(A)=c(d(A))$ for each $A\subset X$.

One of the more noticeable differences between Čech closure spaces and topological spaces is the fact that the composition of two Čech closure operators is a Čech closure operator; but the composition of two Kuratowski or topological closure operators need not be a topological closure operator, but it is a Čech closure operator. Such examples are easy to construct. For finite spaces, this reflects the matrix theory fact that the product of two transitive matrices, with respect to Boolean arithmetic, need not be a transitive matrix.

THEOREM 4.1. *Let (X,c) and (X,d)* be *Čech closure spaces. Then $c\circ d$ is a Čech closure operator on X.*

PROOF. Axioms (1) and (2) are easily satisfied. Let A and B be subsets of X. Then $(c\circ d)(A\cup B)=c(d(A\cup B))=c(d(A)\cup d(B))=c(d(A))\cup c(d(B))=(c\circ d)A\cup(c\circ d)B$.

COROLLARY 4.2. *Let c and d be topological closure operators on X. Then $c\circ d$ and $d\circ c$ are Čech closure operators on X.*

THEOREM 4.3. *Let c and d be Čech closure operators on a set X. Then: $(c\circ d)(A)\supset(c\cup d)(A)$ for each $A\subset X$.*

PROOF. Let $x\in(c\cup d)(A)$. *Then* $x\in c(A)$ or $x\in d(A)$. If $x\in d(A)$ then easily

$x \in c(d(A))$ and thus $x \in (c \circ d)(A)$. If $x \in c(A)$ then $x \in c(d(A))$, since $A \subset d(A)$ and by lemma A, $c(A) \subset c(d(A))$. Thus $x \in (c \circ d)(A)$.

Let (X, c) be a Čech closure space. A natural question arises; is it possible to find a Čech closure operator c^{-1} on X such that $(c^{-1} \circ c)(A) = A$ for each $A \subset X$? However, by theorem 4.3, $(c^{-1} \circ c)(A) \supset c^{-1}(A) \cup c(A)$. Hence it is evident that such a c^{-1} exists if and only if $c(A) = A$ for each $A \subset X$. That is; c^{-1} exists only for the discrete closure operator. This, together with the following theorem implies that it is impossible, except for I, to find inverses for zero-one reflexive matrices with respect to multiplication and Boolean arithmetic.

THEOREM 4.4. *Let c and d be Čech closure operators on a finite set S represented by the matrices T_c and T_d, respectively. Then $T_{c \circ d} = T_d T_c$, where the matrix multiplication is with respect to Boolean arithmetic.*

PROOF. Let $S = \{s_1, s_2, \cdots, s_n\}$. Then the element in the i-th row and the j-th column of $T_d T_c$ is $\bigvee_{k=1}^{n} (t_{ik}^{d} \wedge t_{kj}^{c})$. Now:

$$t_{ij}^{c \circ d} = \begin{cases} 1 & \text{if } s_j \in (c \circ d)(s_i) \\ 0 & \text{if } s_j \notin (c \circ d)(s_i). \end{cases}$$

Thus:

$$t_{ij}^{c \circ d} = \begin{cases} 1 & \text{if there exists } s_k \text{ such that } s_j \in c(s_k) \text{ and } s_k \in d(s_i) \\ 0 & \text{otherwise} \end{cases}$$

and,

$$t_{ij}^{c \circ d} = \begin{cases} 1 & \text{if there exists } k \text{ such that } t_{kj} = 1 \text{ and } t_{ik} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $t_{ij}^{c \circ d} = \bigvee_{k=1}^{n} (t_{ik} \wedge t_{kj})$ and hence $T_{c \circ d} = T_d T_c$.

It is an easy matter to construct examples to show that $c \circ d$ need not equal $d \circ c$. This, for finite spaces, reflects the matrix theory fact that matrix multiplication is not commutative, even with respect to Boolean arithmetic. Even though $c \circ d$ need not equal $d \circ c$, the following theorem yields the rather surprising result that they both generate the same underlying topology.

THEOREM 4.5. *Let c and d be Čech closure operators on X. Then,*

$$t(c \circ d) = t(c) \cap t(d) = t(d \circ c).$$

PROOF. Let $O \in t(c \circ d)$. Then $X - O = (c \circ d)(X - O) = c(d(X - O))$. Now $X - O \subset d(X - O)$ and thus $O \in t(d)$: also $c(X - O) \subset c(d(X - O)) = X - O$. Thus $O \in t(c)$. Hence $t(c \circ d) \subset t(c) \cap t(d)$. Let $O \in t(c) \cap t(d)$. Then $c(X - O)$

$=X-O$ and $d(X-O)=X-O$. Now $(c\circ d)(X-O)=c(d(X-O))=c(X-O)=X-O$. Thus $t(c)\cap t(d)\subset t(c\circ d)$. In a similar manner, $t(d\circ c)=t(c)\cap t(d)$.

COROLLARY 4.6. *Let c and d be Čech closure operators on a set X. Let $\overline{A}$ denote the closure of A with respect to the topology $t(c\circ d)$. Then*:

$\overline{A}=\cap\{F\subset X : A\subset F$ *and F is closed with respect to both* $t(c)$ *and* $t(d)\}$.

THEOREM 4.7. *Let c and d be Čech closure operators on a finite set S. Then* $(c\circ d)^T=d^T\circ c^T$.

PROOF. Let T_c and T_d be the matrices that represent c and d, respectively. Then $T_{(c\circ d)^T}=(T_{c\circ d})^T=(T_dT_c)^T=T_c^T\,T_d^T$. Hence $(c\circ d)^T=d^T\circ c^T$.

EXAMPLE 4.1. $(d\circ c)'$ need not equal either $c'\circ d'$ or $d'\circ c'$. Let $S=\{s_1, s_2, s_3\}$ and c and d Čech closure operators on S represented by the matrices T_c and T_d, respectively; where,

$$T_c=\begin{bmatrix}1&0&1\\1&1&0\\0&0&1\end{bmatrix} \quad\text{and}\quad T_d=\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}.$$

Then:

$$T_{c'}=\begin{bmatrix}1&1&0\\0&1&1\\1&1&1\end{bmatrix} \qquad T_{d'}=\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}.$$

$$T_{d\circ c}=\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix} \qquad T_{(d\circ c)'}=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}.$$

But:

$$T_{d'\circ c'}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \quad\text{and}\quad T_{c'\circ d'}=\begin{bmatrix}1&1&1\\0&1&1\\1&1&1\end{bmatrix}.$$

LEMMA 4.8. *Let (S,c) be a finite Čech closure space. Let $\overline{A}$ denote the closure of A with respect to the underlying topology $t(c)$. Then*:

(1) *For each natural number n, and $A\subset S$, it follows that $c^n(A)\subset\overline{A}$.*

(2) *For each $A\subset S$ there exists a smallest $m\in N$ such that $c^m(A)=\overline{A}$.*

PROOF. (1). $\overline{A}$ is closed and hence $c(\overline{A})=\overline{A}$. Also $A\subset\overline{A}$. Thus $c(A)\subset c(\overline{A})=\overline{A}$. Suppose $c^k(A)\subset\overline{A}$. Then $c^{k+1}(A)\subset c(\overline{A})=\overline{A}$. Hence $c^n(A)\subset\overline{A}$ for each natural number n.

(2). Since S is finite it follows that there exists a smallest natural number m such that $c^m(A)=c^{m+1}(A)$. By (1), $c^m(A)\subset\overline{A}$. Now $c^m(A)$ is closed with respect to $t(c)$ and contains A, hence $c^m(A)=\overline{A}$, since A is the smallest closed set containing A.

DEFINITION 4.2. Let (X,c) be a Čech closure space. Then c is said to *be of degree* k if k is the smallest natural number for which $c^k=c^{k+1}$.

It is apparent that c is a Kuratowski closure operator if and only if it is of degree one. Easily there exist finitely generated Čech closure spaces that are not of finite degree.

THEOREM 4.9. *Let (S,c) be a finite Čech closure space of degree k. Let T denote the matrix representing c. Then*:

(1) *c^k is a Kuratowski closure operator on S.*

(2) *c^k is the closure operator with respect to the underlying topology $t(c)$.*

(3) *The matrix representing $t(c)$, or c^k, is T^k.*

(4) *c^T is a Čech closure operator on S of degree k.*

PROOF. (1) is evident. (2). Let $\overline{A}$ denote the closure of $A\subset S$ with respect to $t(c)$. By lemma 4.8, $c^k(A)\subset\overline{A}$ and there exists a smallest natural number m such that $c^m(A)=\overline{A}$. Suppose $m>k$; then $c^k(A)\neq c^{k+1}(A)$ and we have a contradiction. Thus $k\geq m$ and $c^k(A)=c^m(A)=\overline{A}$.

(3). Since c^k is the closure operator with respect to $t(c)$, it follows from theorem 4.4 that T^k is the matrix that represents c^k and hence represents $t(c)$.

(4). c^T is represented by T^T. Now k is the smallest natural number such that T^k is idempotent. Hence, k is the smallest natural number such that $(T^T)^k$ is idempotent and c^T is of degree k.

5. Semi-topologi s

DEFINITION 5.1. Let X be a non-empty set and $\check{t}\subset P(X)$. Then $\check{t}$ is called *a semi-topology* provided:

(1) $X\in\check{t}$, and $\phi\in\check{t}$

(2) $O\in\check{t}$ and $Q\in\check{t}$ implies $O\cap Q\in\check{t}$.

Let (X,c) be a Čech closure space. A set F is called *c-closed* if there exists $A\subset X$ such that $F=c(A)$. A set O is called *c-open* if $O=X-F$ where F is a c-closed set. $\check{t}(c)$ will denote the collection of all c-open sets in (X,c).

THEOREM 5.1. *Let (X,c) be a Čech closure space. Then $\check{t}(c)$ is a semi-topology.*

PROOF. Easily X and ϕ are c-open sets. Suppose O and Q belong to $t(c)$. Then there exists c-closed sets F and G such that $O=X-F$ and $Q=X-G$. Now there exists subsets A and B such that $F=c(A)$ and $G=c(B)$. Then $F\cup G$ $=c(A)\cup c(B)=c(A\cup B)$, and thus $F\cup G$ is c-closed. Hence $O\cap Q=(X-F)\cap(X-G)=X-(F\cup G)$ is c-open. Thus $\check{t}(c)$ is a semi-topology on X.

COROLLARY 5.2. *Let (X,c) be a Čech closure space. Then $\check{t}(c)\supset t(c)$.*

THEOREM 5.3. *Let (X,c) be a Čech closure space of degree n. Then $\check{t}(c)\supset\check{t}(c^2)\supset\cdots\supset\check{t}(c^n)=t(c)$.*

PROOF. Let $O\in\check{t}(c^{k+1})$. Then there exists sets F and A such that $F=c^{k+1}(A)$ and $O=X-F$. Now $F=c(c^k(A))$ and hence $O\in\check{t}(c^k)$.

EXAMPLE 5.1. Not every semi-topology is generated by a Čech closure operator. Let $X=\{1,2,3,4\}$ and $\check{t}=\{X,\phi,\{1\},\{3\},\{4\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{1,3,4\},\{2,3,4\}\}$. Then $(X,\check{t})$ is a semi-topological space. Suppose $\check{t}=\check{t}(c)$ for some Čech closure operator c on X. Then $\{1,3,4\}=X-c(A)$ for some $A\subset X$. Thus $c(2)=\{2\}$. Similarly, since $\{2,3,4\}\in\check{t}$, $c(1)=\{1\}$. Since $\{2,3\}$ and $\{1,3\}$ belong to $\check{t}$ it follows that $c(4)=\{4\}$. Since $\{1,4\}\in\check{t}$, then $c(3)=\{3\}$ or $c(3)=\{2,3\}$. But $c(3)\neq\{3\}$ since we would then have that $\check{t}$ is the discrete topology.
Thus; if there exists a Čech closure operator c on X such that $\check{t}=\check{t}(c)$ then $c(1)=\{1\}$, $c(2)=\{2\}$, $c(3)=\{2,3\}$ and $c(4)=\{4\}$. Then:
$\check{t}(c)=\{X,\phi,\{1\},\{3\},\{4\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{1,2,3\},\{1,3,4\},\{2,3,4\}\}\neq\check{t}$.
Hence $\check{t}$ is not generated by a Čech closure operator.

EXAMPLE 5.2. Distinct Čech closure operators may generate the same semi-topology. Let $S=\{1,2,3\}$ and let c and d be Čech closure operators on S such that: $c(1)=\{1,3\}, c(2)=\{2,3\}$, and $c(3)=\{2,3\}; d(1)=\{1,3\}, d(2)=\{2,3\}$, and $d(3)=\{1,3\}$. Then $\check{t}(c)=\check{t}(d)$.

THEOREM 5.4. *Let (X,t) be a semi-topological space. Let $\check{c}:P(X)\to P(X)$ be defined by $c(A)=\{x\in X: if\ x\in O\in\check{t}\ then\ O\cap A\neq\phi\}$. Then $\check{c}$ is a Kuratowski closure operator on X.* (The topology generated by $\check{c}$ will be denoted by $t(\check{c})$.)

PROOF. Easily $\check{c}(\phi)=\phi$, and $c(A)\supset A$ for each $A\subset X$. Let A and B be subsets of X and $x\in\check{c}(A\cup B)$. Suppose $x\notin\check{c}(A)\cup\check{c}(B)$. Then there exists c-open sets O and Q containing x such that $O\cap A=\phi$ and $Q\cap A=\phi$. But $x\in O\cap Q\in\check{t}$ and we have a contradiction since $x\in\check{c}(A\cup B)$. Hence $\check{c}(A\cup B)\subset\check{c}(A)\cup\check{c}(B)$. Easily $\check{c}(A)\cup\check{c}(B)\subset\check{c}(AB)$.

Suppose $x \in \check{c}(\check{c}(A))$ for $A \subset X$ and $x \notin c(A)$. Then there exists $Q \subset X$ such that $x \in Q \in \check{t}$ and $Q \cap A = \phi$. Since $x \in \check{c}(\check{c}(A))$ there exists $p \in Q \cap \check{c}(A)$ and this implies that $Q \cap A \neq \phi$. Hence we have a contradiction and thus $\check{c}(\check{c}(A)) = \check{c}(A)$.

THEOREM 5.5. *Let* $(X, \check{t})$ *be a semi-topological space. Then* $t(\check{c}) \supset \check{t}$.

PROOF. Let $O \in \check{t}$. Then $X - O \subset \check{c}(X - O)$. If $x \in \check{c}(X - O)$ then $x \notin O$. Hence $\check{c}(X - O) = X - O$. Therefore $O \in t(\check{c})$.

If (X, c) is a Čech closure space then $\check{t}(c)$ is a semi-topology and it generates a Kuratowski closure operator $\check{c}$. Using this notation we have the following corollary.

COROLLARY 5.6. *Let* (X, c) *be a Čech closure space of degree* n. *Then:* $t(\check{c}) \supset \check{t}(c) \supset \check{t}(c^2) \supset \cdots \supset \check{t}(c^n) = t(c)$.

COROLLARY 5.7. *Let* (X, c) *be a Čech closure space. Let* $A \subset X$. *Then:* $A \subset \check{c}(A) \subset c(A) \subset \overline{A}$.

THEOREM 5.8. *Let* (X, c) *be a Čech closure space. Then:*

(1) $\check{t}(c)$ *is a base for the topology* $t(\check{c})$,

(2) $t(\check{c}) = inf$ $\{t : t$ *is a topology on* X *and* $\check{t}(c) \subset t\}$.

PROOF. (1). Let $O \in t(\check{c})$. Then $\check{c}(X - O) = X - O$. Let $x \in O$. Then there exists $Q_x \in \check{t}(c)$ such that $x \in Q_x$ and $Q_x \cap (X - O) = \phi$. Hence $O = \cup \{Q_x : x \in O\}$.

The proof of (2) follows easily from (1).

6. Separation properties

DEFINITION 6.1. Let (X, c) be a Čech closure space.

(1) (X, c) is called T_0 if for each pair of distinct elements x and y in X then either $x \notin c(y)$ or $y \notin c(x)$.

(2) (X, c) is called T_1 if for each $x \in X$ then $c(x) = \{x\}$.

(3) (X, c) is called *symmetric* (R_0) if $x, y \in X$ and $x \in c(y)$ then $y \in c(x)$.

(4) (X, c) is called *strongly symmetric* if for each $x \in X$ and $A \subset X$ with $c(x) \cap c(A) \neq \phi$ then $x \in c(A)$.

(5) (X, c) is called R_1 if for each $x, y \in X$ then either $c(x) \cap c(y) = \phi$ or $c(x) = c(y)$.

(6) (X, c) is called T_y if for each pair of distinct elements x and y in X then $c(x) \cap c(y)$ is either ϕ or a singleton set.

(7) (X, c) is called T_{ys} if for each pair of distinct elements x and y in X then $c(x) \cap c(y)$ equals either ϕ, $\{x\}$, or $\{y\}$.

(8) (X, c) is called T_F if for each $x \in X$ and disjoint finite set F then either $x \notin c(F)$ or $c(x) \cap F = \phi$.

(9) (X, c) is called T_{FF} if for each pair of disjoint finite sets F and G then either $c(F) \cap G = \phi$ or $F \cap c(G) = \phi$.

A discussion of the origination of several of the above separation axioms for topological spaces is found in Bonnett and Porter [5]. Many of the relationships between the above separation axioms that hold for topological spaces are also valid for Čech closure spaces.

LEMMA 6.1. *The following properties hold for Čech closure spaces.*

(1) $T_1 \Longrightarrow$ *strongly symmetric* $\Longrightarrow R_1 \Longrightarrow$ *Symmetric.*

(2) $T_1 \Longrightarrow T_{ys} \Longrightarrow T_y \Longrightarrow T_0$, *and* $T_{FF} \Longrightarrow T_F \Longrightarrow T_0$.

(3) $T_0 +$ *Symmetric* $\Longleftrightarrow T_1$.

PROOF. All parts are evident except perhaps that strongly symmetric implies R_1. Observe, that independently, strongly symmetric implies symmetric. Now suppose that (X, c) is a strongly symmetric Čech closure space. Let $z \in c(x) \cap c(y)$. Then $x \in c(z)$ and $y \in c(z)$. Let $a \in c(x)$, then $x \in c(a)$. Then $x \in c(y) \cap c(a)$. Hence $c(a) \cap c(y) \neq \phi$, and since (X, c) is strongly symmetric, it follows that $a \in c(y)$. Thus $c(x) \subset c(y)$. Similarly $c(y) \subset c(x)$.

EXAMPLE 6.1. A Čech closure space can be R_1 and not strongly symmetric. Let N denote the natural numbers, and let E denote the set of even natural numbers.

Let $$e(n) = \begin{cases} \{n, n+1\} & \text{if } n \text{ is even} \\ \{n, n-1\} & \text{if } n \text{ is odd.} \end{cases}$$

Define c by:

$$c(A) = \begin{cases} \{e(n) : n \in A\} & \text{if } A \text{ is finite or } \phi \text{ if } A = \phi. \\ \{e(n) : n \in A\} \ E & \text{if } A \text{ contains infinitely many even numbers.} \\ N & \text{otherwise.} \end{cases}$$

Then (N, c) is an R_1 Čech closure space. Let $A = \{2n : n \geq 10\}$. Then $c(3) \cap c(A) \neq \phi$, but $3 \notin c(A)$. Hence (N, c) is not strongly symmetric.

LEMMA 6.2. (1). *For topological spaces; strongly symmetric is equivalent to* R_1. (2). *For finitely generated Čech closure spaces strongly symmetric is equivalent to* R_1.

PROOF. (1). Let (X, t) be a R_1 topological space and $y \in \overline{\{x\}} \cap \overline{A}$. Now $y \in \overline{\{x\}}$

implies $x\in\overline{\{y\}}$. Since $y\in\overline{A}$, it follows that $x\in\overline{\overline{A}}=\overline{A}$. Hence the space is strongly symmetric. Now by lemma 6.1, strongly symmetric is equivalent to R_1.

(2). Let (X,c) be a finitely generated R_1 Čech closure space. Suppose $t\in c(x)\cap c(A)$. Then, since (X,c) is finitely generated, there exists $a\in A$ with $t\in c(a)$. Since the space is R_1 it follows that $c(x)=c(a)$ and $x\in c(A)$. Hence the space is strongly symmetric. The conclusion follows by lemma 6.1.

THEOREM 6.3. *Every finitely generated R_1 Čech closure space is a topological space.*

PROOF. Let (X,c) be such a space and $x\in c(c(A))$. Then there exists a $t\in c(A)$ such that $x\in c(t)$. Now there exists $a\in A$ such that $t\in c(a)$. Since R_1 implies symmetric, it follows that $t\in c(x)$ and $t\in c(a)$. Hence $c(x)=c(a)$ since the space is R_1. Thus $x\in c(A)$ and $c(c(A))=c(A)$ and c is a Kuratowski closure operator.

COROLLARY 6.4. (1) *Every finite R_1 Čech closure space is a topological space.*
(2) *Every finitely generated strongly symmetric Čech closure space is a topological space.*

COROLLARY 6.5. *Let S be a finite set. The following statements are equivalent.*
(1) *(S,c) is a R_1 Čech closure space.*
(2) *(S,c) is a symmetric topological space.*
(3) *(S,c) is a 0-dimensional topological space.*
(4) *(S,c) is a completely regular topological space.*
(5) *(S,c) is a regular topological space.*

PROOF. The proof follows from theorem 6.3 and theorem 3.6 in [5].

THEOREM 6.6. *Every finitely generated T_F Čech closure space is a topological space.*

PROOF. Let (S,c) be such a space. Suppose (S,c) is not a topological space. Then there exists distinct elements i,j,k of S such that $j\in c(i)$ and $k\in c(j)$ but $k\notin c(i)$. Now $j\notin\{i,k\}$. Hence, either $j\notin c\{i,k\}$ or $c(j)\cap\{i,k\}=\phi$. But $j\in c(i)$ and $k\in c(j)$. Therefore (S,c) is a topological space.

COROLLARY 6.7. (1) *Every finite T_F Čech closure space is a topological space.*
(2) *Every finitely generated T_{FF} Čech closure space is a topological space.*

(3) *Every finite T_{FF} Čech closure space is a topological space.*

THEOREM 6.8. *Let (X, c) be a finitely generated symmetric Čech closure space. Then $O \in t(c)$ if and only if $X-O \in t(c)$.*

PROOF. Let $O \in t(c)$. Then $X-O=c(X-O)$. Let $x \in c(O)$ and since (X, c) is finitely generated, there exists $p \in O$ with $x \in c(p)$. Suppose $x \in X-O$. Then, since (X, c) is R_0, we have that $p \in c(x)$ and thus $p \in c(X-O)=X-O$, a contradiction. Thus, $c(O)=O$ and $X-O \in t(c)$.

COROLLARY 6.9. *In a finite symmetric topological space a set is open if and only if it is closed.*

We now consider finite Čech closure spaces and characterize various separation axioms in terms of the matrices that represent the Čech closure operators.

THEOREM 6.10. *Let (S, c) be a finite Čech closure space and T_c the matrix that represents c. Then the following pairs of statements are equivalent.*

(*A*) *(S, c) is a symmetric Čech closure space.*

(*A'*) *T_c is a symmetric matrix.*

(*B*) *(S, c) is a R_1 Čech closure space.*

(*B'*) *Two rows of T_c are either equal or disjoint.*

(*C*) *(S, c) is a strongly symmetric Čech closure space.*

(*C'*) *Two rows of T_c are either equal or disjoint.*

(*D*) *(S, c) is a T_0 Čech closure space.*

(*D'*) *T_c is anti-symmetric.*

(*E*) *(S, c) is a T_1 Čech closure space.*

(*E'*) *$T_c = I$.*

(*F*) *(S, c) is a T_y Čech closure space.*

(*F'*) *$(T_c)_i \wedge (T_c)_j$, for $i \neq j$, has at most one non-zero element.*

(*G*) *(S, c) is a T_{ys} Čech closure space.*

(*G'*) *$(T_c - I)(T_c - I)^T$, with respect to the usual matrix multiplication, is a diagonal matrix.*

(*H*) *(S, c) is a T_F Čech closure space.*

(*H'*) *For each i, either the i-th row or the i-th column of $T_c - I$ is zero.*

(*I*) *(S, c) is a T_{FF} Čech closure space.*

(*I'*) *$(T-I)$or $(T-I)^T$ has at most one non-zero row.*

PROOF. The proofs of (*A*) through (*G*) are easy and omitted. By corollary

6.7, the spaces in (H) and (I) are topological spaces and the results follow from theorem 3.2 and theorem 3.5 in [5], respectively.

COROLLARY 6.11. *Let (S,c) be a finite Čech closure space. Then:*

(1) (S,c) *is* $T_0 \Longleftrightarrow (S,c^T)$ *is* T_0.

(2) (S,c) *is* $T_1 \Longleftrightarrow (S,c)$ *is a discrete topological space.*

(3) (S,c) *is* $T_1 \Longleftrightarrow (S,c^T)$ *is* T_1.

(4) (S,c) *is symmetric* $\Longleftrightarrow (S,c^T)$ *is symmetric.*

(5) (S,c) *is symmetric* $\Longleftrightarrow c=c^T$.

(6) (S,c) *is* $T_y \Longleftrightarrow (S,c^T)$ *is* T_y.

(7) (S,c) *is* $T_{ys} \Longleftrightarrow (S,c^T)$ *is* T_{ys}.

(8) (S,c) *is* $T_F \Longleftrightarrow (S,c^T)$ *is* T_F.

(9) (S,c) *is* $T_{FF} \Longleftrightarrow (S,c^T)$ *is* T_{FF}.

It is shown in [5] that R_1 is equivalent to R_0 (symmetric) for finite topological spaces. That this is not the case for finite Čech closure spaces follows easily from the fact that there exists reflexive and symmetric zero-one matrices that are not transitive.

It is of interest to know which separation properties of a Čech closure space are preserved by the underlying topology. The next theorem and the following examples examine this question.

THEOREM 6.12. *Let (X,c) be a finitely generated Čech closure space.*

(1) *If (X,c) is T_1 then $t(c)$ is T_1.*

(2) *If (X,c) is T_F then $t(c)$ is T_F.*

(3) *If (X,c) is T_{FF} then $t(c)$ is T_{FF}.*

(4) *If (X,c) is R_1 then $t(c)$ is R_1.*

(5) *If (X,c) is strongly symmetric then $t(c)$ is strongly symmetric.*

PROOF. (1) is evident. The remaining parts follow from the fact that the stated separation axioms are strong enough to assure that (X,c) is actually a topological space, as shown by the previous results.

EXAMPLE 6.2. A finitely generated Čech closure space may be T_0 and the underlying topology not be T_0. Let $S=\{s_1,s_2,s_3\}$ and let c be represented by the matrix

$$T_c = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}.$$

Then (X,c) is T_0 and $t(c)$ is the trivial topology on S.

EXAMPLE 6.3. The underlying topology of a symmetric Čech closure space need not be symmetric. Let N denote the natural numbers. Let:

$$e(n) = \begin{cases} \{n\} \text{ if } n \text{ is even} \\ \{1, n\} \text{ if } n \text{ is odd and } n \neq 1 \\ \text{odd natural numbers if } n = 1. \end{cases}$$

Define:

$$c(A) = \begin{cases} \phi \text{ if } A = \phi \\ \bigcup \{e(a) : a \in A\} \text{ if } A \text{ is finite} \\ N \text{ if } A \text{ is infinite.} \end{cases}$$

Then (N, c) is a symmetric Čech closure space. $\overline{\{2\}} = \{2\}$ since $c(2) = \{2\}$. But $\overline{\{1\}} = c(c(1)) = N$ and thus $t(c)$ is not symmetric.

EXAMPLE 6.4. A finitely generated Čech closure space may be T_{ys} (and hence T_y) and the underlying topology not be T_y and thus not T_{ys}. Let $S = \{s_1, s_2, s_3\}$ and c be the Čech closure operator represented by the matrix

$$T_c = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Easily (S, c) is T_{ys} and hence T_y. $t(c)$, however, is the trivial topology and thus neither T_{ys} nor T_y.

THEOREM 6.13. *Let (X, c) be a finitely generated symmetric Čech closure space of finite degree. Then the underlying topology is symmetric.*

PROOF. Let (X, c) be a finitely generated symmetric Čech closure space of degree n. Suppose $x \in \overline{\{a\}} = c^n(a)$. Since the space is finitely generated there exists $x_i \in X$ such that $x \in c(x_1)$, $x_2 \in c(x_3)$, $\cdots$, $x_n \in c(a)$. Since the space is symmetric; $x_1 \in c(x)$, $x_3 \in c(x_2)$, $\cdots$, $a \in c(x_n)$. Thus $a \in c^n(x) = \overline{\{x\}}$. Hence $t(c)$ is symmetric.

The following theorem summarizes results obtained in this paper and also results obtained by Sharp [12].

They are listed together in order to unify the concepts. The proof clearly denotes which of the following statements are due to Sharp.

THEOREM 6.14. *Let S be a finite set. Then:*

(1) *The reflexive relations on S correspond to the Čech closure operators on S.*

(2) *The reflexive and symmetric relations on S correspond to the symmetric Čech closure operators on S.*

(3) *The reflexive, antisymmetric relations on S correspond to the T_0 Čech closure operators on S.*

(4) *The reflexive and transitive relations, (quasi-orders) on S correspond to the topologies on S.*

(5) *The reflexive, transitive and anti-symmetric relations (partial-orders) on S correspond to the T_0 topologies on S.*

(6) *The equivalence relations on S correspond to the symmetric topologies on S.*

PROOF. Statements (4), (5), and (6) are due to Sharp [12]. Statements (1), (2), and (3) are easy consequences of results obtained in this paper.

Davis, in [8], provides a technical formula for the number of non-isomorphic reflexive relations on a set of n-elements. Consequently, this is also the number of non-homeomorphic Čech closure operators on a set on n-elements.

7. Categorical implications

THEOREM 7.1. *The category of topological spaces and continuous maps is bi-reflective in the category of Čech closure spaces of finite degree and continuous maps.*

PROOF. The reflection is given by $i:(X,c)\rightarrow(X,t(c))$. Since $c(A)\subset\overline{A}$, i is continuous. Consider the following diagram:

$$\begin{array}{ccc} (X,c) & \xrightarrow{\ i\ } & (X,t(c)) \\ f \big\downarrow & & \big\downarrow \hat{f} \\ & \longrightarrow & (Y,d) \end{array}$$

Where (X,c) is a Čech closure space of degree n, and (Y,d) is a topological space. For each $A\subset X$, $\overline{A}$ denotes the closure of A with respect to $t(c)$. Let $\hat{f}(x)=f(x)$ for each $x\in X$. Easily $\hat{f}$ is unique.

To see that $\hat{f}$ is continuous, let $A\subset X$. Since f is continuous then $f(c(A))\subset d(f(A))$. Then:

$$\hat{f}(\overline{A})=f(c^n(A))\subset d^n(f(A))=d(f(A))=d(\hat{f}(A)).$$

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