The purpose of this paper is to prove that in a manifold of dimension \( \geq 2 \), for any two sets \( A \) and \( B \) having no cluster points and having same cardinality, then there exists homeomorphism \( f \) such that \( f(A) = B \). Then we use this property to study the topology \( \mathcal{Y} \) with the same class of homeomorphisms \( H(Z, \mathcal{Y}) = H(X, \mathcal{Y}) \). By manifold we always mean separable connected manifold without boundary.

**THEOREM 1.** Let \( X \) be a manifold with dimension \( \geq 2 \) and \( A, B \) are subsets of \( X \) with no cluster point and have same cardinality. Then there exists a homeomorphism \( f \) of \( X \) onto itself such that \( f(A) = B \).

**PROOF.** Since \( A \) and \( B \) have no cluster points, the cardinality of \( A \) and \( B \) is at most countable. If \( A \) and \( B \) have same finite number of points. Then by the homogeneity of \( X \), we have a homeomorphism \( f \) of \( X \) onto itself such that \( f(A) = B \). If \( A \) and \( B \) are countable, let \( A = \{a_1, a_2, \ldots, a_n, \ldots\} \)
and \( B = \{b_1, b_2, \ldots, b_n, \ldots\} \).

First choose an open connected set \( U_1 \) such that \( \{a_1, b_1\} \subseteq U_1 \) and \( \text{cl}(U_1) \cap \{a_2, \ldots, a_n, \ldots\} \cup \{b_1, \ldots, b_n, \ldots\} = \emptyset \) and \( X \setminus U_1 \) is a connected manifold and a homeomorphism \( f_1 \) with \( f_1(a_1) = b_1 \) and \( f_1(x) = x \) for \( x \in U_1 \). After constructing \( \{U_{n-1}\} \), choose \( U_n \) to be an open connected set such that
\[
\text{cl}(U_n) \cap \text{cl}(U_1) \cup \ldots \cup \text{cl}(U_{n-1}) \cup \{a_{n+1}, a_{n+2}, \ldots\} \cup \{b_2, \ldots, b_n, \ldots\} = \emptyset
\]
and \( X \setminus \{\text{cl}(U_i)|i=1, 2, \ldots, n\} \) is a connected manifold. In this way we constructed a sequence of open connected sets and sequence of homeomorphism \( \{f_n\} \) such that \( f_n \) is fixed outside \( U_n \) and \( f_n(a_n) = b_n \). Then let \( f \) be the function defined on \( X \) such that \( f(x) = f_n(x) \) if \( x \in U_n \) and \( f(x) = x \) if \( x \notin U_n \) for all \( n \).

Then \( f \) is a homeomorphism and \( f(A) = B \).

This result is useful in studying the classes of homeomorphisms.

**THEOREM 2.** Let \( (Z, \mathcal{Y}) \) be a manifold and \( H(Z, \mathcal{Y}) \) the class of homeomorphisms of \( (X, \mathcal{Y}) \) onto itself. Let \( \mathcal{Y} = \{U \subseteq \mathcal{Z} | U = \emptyset \text{ or } X \setminus U \text{ has no cluster point.}\} \)
then $\mathcal{Y}$ is a topology in $X$ and $H(X, \mathcal{Y}) \subseteq H(X, \mathcal{Y}')$

PROOF. It is clear that $\mathcal{Y}$ is a topology and $H(X, \mathcal{Y}) \subseteq H(X, \mathcal{Y}')$. To see that $H(X, \mathcal{Y}) \neq H(X, \mathcal{Y}')$, take a point $p \in X$ and an open ball $U$ with center $p$. Let $f$ be a function which makes a rotation on $U$ on any direction of angle between $0$ and $\pi$, and fixed outside $U$, then $f \in H(X, \mathcal{Y}')$ but $f \notin H(X, \mathcal{Y})$.

COROLLARY. Let $(X, \mathcal{Y})$ be a manifold and $\mathcal{Y} \subseteq \mathcal{U}$ is a topology on $X$. If $H(X, \mathcal{Y}) \neq H(X, \mathcal{Y}')$, then there exists $V \neq \emptyset$ in $\mathcal{Y}$ such that $X \setminus V$ contains cluster points.

PROOF. Let $\dim(X, \mathcal{Y}) \geq 2$. Then there exists $\phi \neq V \in \mathcal{Y}$ with $X \setminus V$ containing infinitely many points. Because otherwise, by theorem 1 $\mathcal{Y} = \{V \in \mathcal{Y} | V = \emptyset \text{ or } \text{Card}(X \setminus V) \leq m\}$ for some positive integer $m$. Then $H(X, \mathcal{Y})$ would be the set of all one-to-one functions of $X$ onto itself. If $X \setminus V$ contains no cluster points for every non-void $V$ in $\mathcal{Y}$ then by theorem 1 again $\mathcal{Y} = \{V \in \mathcal{Y} | V = \emptyset \text{ or } \text{Card}(X \setminus V) \leq N\}$, and $X \setminus V$ has no cluster points. By theorem 2, $H(X, \mathcal{Y}) \supseteq H(X, \mathcal{Y}')$.

If $\dim(X, \mathcal{Y}) = 1$, and $X$ is a circle, and for every non-void $V \in \mathcal{Y}$ $X \setminus V$ contains no cluster point, then $X \setminus V$ is a finite set, it is easy to see $\mathcal{Y} = \{V \in \mathcal{Y} | V = \emptyset \text{ or } \text{Card}(X \setminus V) \leq m\}$ for some positive integer $m$ and $H(X, \mathcal{Y}) \supseteq H(X, \mathcal{Y}') = \text{the set of all one-one onto maps}$. Hence there is $V \neq \emptyset$ with $X \setminus V$ containing infinitely many points. Since $X$ is compact, $X \setminus V$ has cluster points. If $X$ is a real line, by similar argument, there exists $V \neq \emptyset$ in $\mathcal{Y}$ such that $\text{Card}(X \setminus V) = \mathbb{N}$. If $X \setminus V$ has no cluster point and $X \setminus V$ is unbounded in both sides, then it is easy to see that there exist $f, g \in H(X, \mathcal{Y})$ with $f(X \setminus V) = \text{the set of all integers}$. Also there exist $g \in H(Z, \mathcal{Y})$ such that $g(X \setminus V) = \{1, 2, 3, \ldots\} \cup \left\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\right\}$

Hence $f(X \setminus V) \cap g(Z \setminus V) = \text{the set of all positive integers which is closed } (Z, \mathcal{Y})$. If $Z \setminus V$ is unbounded in one side, then there exist $f, g$ in $H(Z, \mathcal{Y})$ such that $f(X \setminus V) = \text{the set of all non-negative integers}$ $g(Z \setminus V) = \text{the set of all non-positive integers}$. Hence the set of all integers is closed in $(X, \mathcal{Y})$ and $\mathcal{Y} \supseteq \{V \in \mathcal{Y} | V = \emptyset \text{ or } X \setminus V \text{ has no cluster points}\}$.

However, if $\mathcal{Y}$ does not contain any non-void $V$ with $X \setminus V$ having cluster points then $H(X, \mathcal{Y}) \subseteq H(X, \mathcal{Y}')$. This contraction proves that there exists $V \neq \emptyset$ in $\mathcal{Y}$ with $X \setminus V$ contains cluster points.