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ON FIXED POINT THEOREMS IN COMPACT METRIC SPACES

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The results of this paper are inspired by two recent papers of Fisher [2] and Khan [3]. They proved that a continuous mapping T of a compact metric space (X, d) has a unique fixed point if T satisfies

$$d(Tx, Ty) < \frac{1}{2}(d(x, Ty)+d(y, Tx)),$$

or

$$d(Tx, Ty) < (d(x, Tx)d(y, Ty))^{\frac{1}{2}} \text{ for all } x, y \text{ in } X \text{ with } x \neq y.$$

The main purpose of this paper is to extend their results to a more general case. For related results, we refer to Ciric [1] and Yeh [4].

THEOREM 1. *Let T be a continuous mapping of a nonempty compact metric space (X, d) satisfying*

$$(C_1) \qquad d(Tx, Ty) < \max\{d(x, y), \frac{1}{2}(d(x, Tx)+d(y, Ty)), \frac{1}{2}(d(x, Ty)+d(y, Tx)), (d(x, y))^{-1}d(x, Tx)d(y, Ty), a(x, y)d(x, Ty)d(y, Tx), (d(x, Tx)d(y, Ty))^{\frac{1}{2}}, b(x, y)(d(y, Tx)d(x, Ty))^{\frac{1}{2}}\}$$

for all x, y in X with $x \neq y$, where $a(x, y)$ and $b(x, y)$ are nonnegative real functions, then T has a fixed point. If in addition $a(x, y) \leq (d(x, y))^{-1}$ and $b(x, y) \leq 1$, then T has a unique fixed point.

PROOF. Define a real valued function f on X by $f(x)=d(x, Tx)$. Since d and T are continuous functions, it follows that f is a continuous function on X. Since X is compact, it attains its minimum value and so there is a point u in X such that $f(u)=\inf\{f(x) : x \in X\}$. If $u \neq Tu$, then it follows from (C_1) that

$$d(Tu, T^2u) < \max\{d(u, Tu), \frac{1}{2}(d(u, Tu)+d(Tu, T^2u)), \frac{1}{2}(d(u, T^2u)+d(Tu, Tu)), (d(u, Tu))^{-1}d(u, Tu)\, d(Tu, T^2u),$$

$$a(u, Tu)\, d(u, T^2u)d(Tu, Tu),\ (d(u, Tu)d(Tu, T^2u))^{\frac{1}{2}},$$
$$b(u, Tu)\ (d(u, T^2u)d(Tu, Tu))^{\frac{1}{2}}\}$$
$$\leq \max\{d(Tu, T^2u),\ d(Tu, T^2u),\ d(Tu, T^2u),\ d(Tu, T^2u),$$
$$0,\ d(Tu, T^2u),\ 0\} = d(Tu, T^2u)$$

a contradiction. This contradition proves that $Tu=u$.

Next we prove that u is unique for $a(x, y)\leq(d(x, y))^{-1}$ and $b(x, y)\leq 1$. Suppose that $v(\neq u)$ is a fixed point of T. Then

$$d(u, v)=d(Tu, Tv)<\max\{d(u, v),\ 0,\ d(u, v),\ 0,\ d(u, v),\ 0,\ d(u, v)\}$$
$$=d(u, v)$$

a contradiction. This contradiction proves the uniqueness of u. Thus our proof is complete.

THEOREM 2. *Let T be selfmapping of a metric space (X, d) satisfying*

$$(C_2)\qquad d(Tx, Ty)\geq \max\{d(x, y),\ \frac{1}{2}(d(x, Ty)+d(y, Tx)),$$
$$\frac{1}{2}(d(x, Tx)+d(y, Ty)),\ (d(x, Tx)\, d(y, Ty))^{\frac{1}{2}},$$
$$(d(y, Tx)d(x, Ty))^{\frac{1}{2}}\}$$

for all x, y in X. Then T is the identity mapping on X.

PROOF. For any point x in X, we have

$$0=d(Tx, Tx)\geq \max\{d(x, x),\ \frac{1}{2}(d(x, Tx)+d(x, Tx)),$$
$$\frac{1}{2}(d(x, Tx)+d(x, Tx)),\ (d(x, Tx)\, d(x, Tx)^{\frac{1}{2}},$$
$$(d(x, Tx)d(x, Tx))^{\frac{1}{2}}\}=d(x, Tx).$$

Hence $d(x, Tx)=0$ or $Tx=x$. This proves our theorm.

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