ON FINSLER SPACE OF RECURRENT CURVATURE TENSORS

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0. Summary

The Riemannian space of recurrent curvature was defined and studied by Ruse [8] and Walker [10]. In 1963, Móor [4] generalised this idea for Finsler spaces and defined and studied Finsler spaces of recurrent curvature. These spaces for various curvature tensors have subsequently been studied by Mishra and Pande [1], Sen [9] and Misra [3] etc. The purpose of the present paper is to study Finsler space based on the recurrency of the curvature tensors derived from non-linear connections.

1. Introduction

Let $X(x^k)$ and $Y(x^k)$ be two differentiable vector fields in a Finsler space $F_n$ with metric tensor $g_{ij}(x, X)$ and non-linear connections $\Gamma^1_{jk}(x, X)$ and $\Gamma^2_{jk}(x, Y)$, positively homogeneous of first degree in $X$ and $Y$ respectively, then we have Rund [6]:

\begin{equation}
\Gamma^i_{jk} = \Delta_j \Gamma^i_k, \quad X^j \Gamma^i_{jk} = \Gamma^i_j
\end{equation}

and

\begin{equation}
\Gamma^2_{jk} = \Delta_i \Gamma^2_{jk}, \quad Y^j \Gamma^2_{jk} = \Gamma^2_{jk},
\end{equation}

where $\Delta_j = \partial/\partial x^j$ and $\Delta^i = \partial/\partial Y^i$.

Let us suppose that if $X^i$ undergoes a parallel displacement then so does $Y^i = g_{ij}X^j$, such that the length of a vector remains unchanged under parallel displacement, then we have [6]:

\begin{equation}
2G^i = \Gamma^i_kX^k + g^{ik}Y^j(\Gamma^j_{hk}X^k - \Gamma^i_k).
\end{equation}

Assuming geodesics to be auto-parallel curves of $F_n$ we get

\begin{equation}
2G^i = \Gamma^i_kX^k.
\end{equation}
such that

\[
\frac{1}{2} I_{hj} = G_{hj} + \frac{1}{2} \left( S_{hj} - X^k \Delta_j S_{kh} \right)
\]

and

\[
\frac{2}{h} I_{tk} = \frac{1}{h} + Y^k \Delta^l I_{tk}.
\]

where

\[
\frac{1}{h} S_{hh} = 2 I_{[hh]}.
\]

The covariant derivative of a tensor \( T^i_j (x, X) \) is defined by [2]:

\[
T^i_j (x, X) = \partial_k T^i_j + (\Delta_m T^i_j) (\partial_k X^m) + T^m_j \frac{I_{mk}}{I_{kk}} - T^m_i \frac{I_{km}}{I_{kk}}.
\]

The two curvature tensors based on these coefficients of connections are given by [2]:

\[
R^i_{jkh} (x, X) = 2 \left[ \partial_k^i \frac{I_{j[k]}^i}{I_{j[k]}} + (\Delta_m \frac{I_{j[m]}^i}{I_{j[m]}}) (\partial_k^i X^m) + \frac{I_{m[j]}^i}{I_{j[j]}} \frac{I_{m[i]}}{I_{m[i]}} \right]
\]

and

\[
R^i_{jkh} (x, Y) = 2 \left[ \partial_k^i \frac{I_{j[k]}^i}{I_{j[k]}} + (\Delta_m \frac{I_{j[m]}^i}{I_{j[m]}}) (\partial_k^i Y^m) + \frac{I_{m[j]}^i}{I_{j[j]}} \frac{I_{m[i]}}{I_{m[i]}} \right]
\]

and satisfy

\[
X^i R^1_{jkh} = R^1_{kh} \quad Y^i R^2_{jkh} = R^2_{kh}.
\]

such that

\[
R^1_{jkh} (x, X) = \Delta^i \frac{I_{j[i]}^i}{I_{j[i]}} + 2 (\Delta_m \frac{I_{j[m]}^i}{I_{j[m]}}) X^m_{i[h]}
\]

and

\[
R^2_{jkh} (x, Y) = \Delta^i \frac{I_{j[i]}^i}{I_{j[i]}} + 2 (\Delta_m \frac{I_{j[m]}^i}{I_{j[m]}}) Y^m_{i[h]}
\]

2. Recurrent curvature tensors

DEFINITION 2.1. If in a non-flat Finsler space \( F_n \) the curvature tensors \( R^1_{jkh}, R^2_{jkh} \), \( R^1_{kh} \) and \( R^2_{jkh} \) satisfy

\[
\begin{align*}
\frac{1}{R^1_{jkh}} & = \lambda R^1_{jkh} \quad \frac{2}{R^2_{jkh}} = \lambda R^2_{jkh} \quad \frac{1}{R^1_{kh}} \quad \frac{2}{R^2_{kh}} = \lambda R^2_{kh} \quad \frac{1}{R^1_{kh}} \quad \frac{2}{R^2_{kh}} = \lambda R^2_{kh}
\end{align*}
\]
and

\[(2.1)d \quad 2 \tilde{R}_{jkh,l} = \lambda_i \tilde{R}_{jkh},\]

respectively, for a non-null covariant vector \(\lambda_i\), then they are called \(R\)-recurrent curvature tensors of various types respectively.

**DEFINITION 2.2.** If \(R_{jki} \overset{1}{=}, \tilde{R}_{jki} \overset{2}{=}, \tilde{R}_{jk} \overset{1}{=}, \tilde{R}_{jk} \overset{2}{=}, \tilde{R}_{ij} \overset{1}{=}, \tilde{R}_{ij} \overset{2}{=}, \tilde{R}_{ijk} \overset{1}{=}, \tilde{R}_{ijk} \overset{2}{=}, \tilde{R}_{jkh} \overset{1}{=}, \tilde{R}_{jkh} \overset{2}{=}, \tilde{R}_{jkh} \overset{1}{=}, \tilde{R}_{jkh} \overset{2}{=}, \tilde{R}_{jkh} \overset{1}{=}, \tilde{R}_{jkh} \overset{2}{=}\)

\[1 R_{jkh} \overset{1}{=}, g_{ir} R_{jkh} \overset{2}{=}, 2 R_{jkh} \overset{1}{=}, g_{ir} R_{jkh} \overset{2}{=}, 1 R_{khr} \overset{1}{=}, g_{ir} R_{khr} \overset{2}{=}, 2 R_{khr} \overset{1}{=}, g_{ir} R_{khr} \overset{2}{=}, \]

satisfy

\[(2.2)a \quad 1 \tilde{R}_{jkh,l} = \lambda_i \tilde{R}_{jkh},\]
\[(2.2)b \quad 2 \tilde{R}_{jkh,l} = \lambda_i \tilde{R}_{jkh},\]
\[(2.2)c \quad 1 \tilde{R}_{jk,l} = \lambda_i \tilde{R}_{jk,l},\]
\[(2.2)d \quad 2 \tilde{R}_{jk,l} = \lambda_i \tilde{R}_{jk,l},\]
\[(2.2)e \quad 1 \tilde{R}_{jkh,l} = \lambda_i \tilde{R}_{jkh,l},\]
\[(2.2)f \quad 2 \tilde{R}_{jkh,l} = \lambda_i \tilde{R}_{jkh,l},\]
\[(2.2)g \quad 1 \tilde{R}_{khr,l} = \lambda_i \tilde{R}_{khr,l},\]

and

\[(2.2)h \quad 2 \tilde{R}_{jk,l} = \lambda_i \tilde{R}_{jk,l},\]

respectively, then the various curvature tensors given in the definition are called \(R\)-recurrent curvature tensors respectively.

Multiplying equations (2.1)a and (2.1)b by \(X^j\) and \(Y^l\) respectively we get on simplification

\[(2.3)a \quad 1 \tilde{R}_{kh,l} = \lambda_i \tilde{R}_{kh,l} - X^j \tilde{R}_{jkh},\]

and

\[(2.3)b \quad 2 \tilde{R}_{jkh,l} = \lambda_i \tilde{R}_{jkh,l} - Y^j \tilde{R}_{jkh},\]

which by virtue of equations (2.1)c and (2.1)d imply;
THEOREM 2.1. If \( R_{jkhh}^{1} \) and \( R_{jkhh}^{2} \) are R-recurrent curvature tensors, then the necessary and sufficient conditions for \( R_{jkhh}^{1} \) and \( R_{jkhh}^{2} \) to be R-recurrent are given by \( X_{j}^{i}, R_{jkhh}^{1} = 0 \) and \( Y_{i}, R_{jkhh}^{2} = 0 \), respectively.

Differentiating (2.1)c and (2.1)d partially with respect to \( X^{j} \) and \( Y_{i} \) respectively and using equations (1.11)\( a \) and (1.11)\( b \) we obtain on simplification

\[
(2.4)a \quad R_{jkhh}^{1}, i - \lambda_{i} R_{jkhh}^{1} = (\Delta_{j} R_{jkhh}^{1}) R_{jkhh}^{1} - \Delta_{j} (\Delta_{l} R_{jkhh}^{1}) X_{m}^{m} \\
- (\Delta_{m} R_{jkhh}^{1}) j_{l} - R_{jkhh}^{1} \Delta_{j} j_{l} + R_{mhh} \Delta_{j} j_{l} \\
+ \lambda_{i} R_{jkhh}^{1} \Delta_{j} j_{l} + 2 (\Delta_{j} R_{jkhh}^{1}) X_{m}^{m},.i - (\Delta_{m} R_{jkhh}^{1}) \Delta_{j} (\partial_{i} X^{m})
\]

\[
(2.4)b \quad R_{jkhh}^{2}, i - \lambda_{i} R_{jkhh}^{2} = (\Delta^{l} R_{jkhh}^{2}) R_{jkhh}^{2} - \Delta_{j} (\Delta^{l} R_{jkhh}^{2}) Y_{m}^{m} \\
+ 2 (\Delta^{l} R_{jkhh}^{2}) Y_{m}^{m},.i - (\Delta_{m} R_{jkhh}^{2}) \Delta_{j} (\partial_{i} Y^{m}) \\
+ (\Delta_{m} R_{jkhh}^{2}) j_{l} + R_{mhh} \Delta_{j} j_{l} \\
+ \lambda_{i} R_{jkhh}^{2} \Delta_{j} j_{l} + R_{jkhh}^{2} \Delta_{j} j_{l}
\]

respectively. From equations (2.4)a and (2.4)b by virtue of equations (2.1)a and (2.1)b we obtain;

THEOREM 2.2. If \( R_{jkhh}^{1} \) and \( R_{jkhh}^{2} \) are R-recurrent curvature tensors then the necessary and sufficient condition for \( R_{jkhh}^{1} \) and \( R_{jkhh}^{2} \) to be R-recurrent is given by the vanishing of the right hand side of (2.4)a and (2.4)b respectively.

Multiplying equations (2.1)a and (2.1)b by \( g_{ir} \) we get

\[
(2.5)a \quad R_{jkhr,l}^{1} = \lambda_{l} R_{jkhr}^{1} + g_{ir,l} R_{jkhh}^{1}
\]

and

\[
(2.5)b \quad R_{jkhr,l}^{2} = \lambda_{l} R_{jkhr}^{2} + g_{ir,l} R_{jkhh}^{2}
\]

which by definition (2.2) and equations (2.2)e and (2.2)f lead to;

THEOREM 2.3. If \( R_{jkhh}^{1} \) and \( R_{jkhh}^{2} \) are R-recurrent tensors then their associates will be R-recurrent iff \( g_{ir,l} R_{jkhh}^{1} = 0 \) and \( g_{ir,l} R_{jkhh}^{2} = 0 \), respectively.
Differentiating relations $R_{jk}^h = g^{hr} 1 R_{jkh}^r$ and $R_{jkh}^r = g^{hr} 2 R_{jkhr}^r$ with respect to $x^i$ covariantly and using (2.2)a, (2.2)b and (2.2)e, (2.2)f we obtain

$R_{jkhr} = 0$

which leads to;

**THEOREM 2.4.** If any two of the following are satisfied:

i) $R_{jk}$ is $R$-recurrent ($R_{jk}$ is $R$-recurrent),

ii) $R_{jkhr}$ is $R$-recurrent ($R_{jkhr}$ is $R$-recurrent),

iii) $g^{hr} 1 R_{jkhr} = 0$ ($g^{hr} 2 R_{jkhr} = 0$),

then the third is also satisfied.

**REMARK.** A similar theorem can be established for $R_{j}^k$ and $R_{j}^k$.

### 3. Some special cases

If we consider the covariant differentiation due to Berwald of a tensor $T^i_j (x, X)$ and denote it by $T^i_j (x, X)$, Rund [7], then we can easily establish the following:

$$T^i_{j,h} = T^i_{j(k)} + (\Delta_m T^i_j)(\partial_h x^m - \Delta_h G^m)$$

$$+ \frac{1}{2} T^i_j \{ S^i_{mk} - X^k \Delta_h S^i_{mk} \}$$

$$- T^i_m \{ Y^m_p \Delta_j h^i + \frac{1}{2} (S^m_{jh} - X^h \Delta_h S^m_{jh}) \}.$$ 

Since we know that [5]:

$$R^i_{kh} = 2(\Pi^i_{kh} - M^i_{kh}),$$

where

$$M^i_{kh} = (\partial_{[k} \Pi^i_{h]} (x^j) + G^i_{[k|m]} \Pi^m_{h]} (x^j + \Pi^m_{h]} - \Pi^m_{k]}),$$

therefore by virtue of (3.1) and (3.2) we obtain on simplification
(3.4) \[ \mathbf{R}_{kh,j}^{i} = 2[H_{kh(j)}^{i}] + (\Delta_{m}H_{kh}^{i}) (\partial_{j}X^{m} - \Delta_{j}G^{m}) \\
+ \frac{1}{2}H_{kh}^{i}(S_{mj} - X^{p} \Delta_{j} \dot{S}_{mp}^{i}) \\
- H_{nm}^{i}(Y_{p} \Delta_{m} \dot{S}_{kj}^{i}) + \frac{1}{2}(S_{mj} - X^{p} \Delta_{j} \dot{S}_{mp}^{i})} \\
- H_{nh}^{i}(Y_{p} \Delta_{m} \dot{S}_{kj}^{i}) + \frac{1}{2}(S_{mj} - X^{p} \Delta_{j} \dot{S}_{mp}^{i}) \\
- M_{kh,j}^{i}. \]

Now applying equation (2.1)c and the fact that Finsler space \( F_{n} \) is H-recurrent, i.e., it satisfies \( H_{kh}^{i} = \lambda_{j}H_{kh}^{i} \), we obtain:

**THEOREM 3.1.** If the tensor \( R_{kh}^{i} \) is R-recurrent and \( H_{kh}^{i} \) is H-recurrent, then the necessary and sufficient condition for \( M_{kh}^{i} \) to be R-recurrent is given by

\[ (\Delta_{m}H_{kh}^{i})(\partial_{j}X^{m} - \Delta_{j}G^{m}) + H_{km}^{i}\dot{g}_{mj} - H_{kh}^{i}\dot{g}_{mj} - H_{nh}^{i}\dot{g}_{kj} \\
- H_{nh}^{i}G_{mj}^{i} + H_{km}^{i}G_{hj}^{i} + H_{mh}^{i}G_{kj}^{i} = 0 \]

Since we know that [5]:

(3.5) \[ \mathbf{R}_{jkh}^{i} = 2[H_{jkh}^{i} - M_{jkh}^{i}], \]

where

(3.6) \[ M_{jkh}^{i} = \{\Delta_{j}[\dot{G}_{m[k]}^{i}X^{m}] - (\Delta_{j}[\dot{g}_{m[k]}^{i}X^{m}] - \dot{g}_{mj}^{i}[\dot{g}_{m[k]}^{i}X^{m}] - \dot{g}_{mj}^{i}X^{m} - \dot{g}_{mj}^{i}X^{m} - \dot{g}_{mj}^{i}X^{m} + \dot{G}_{m[k]}^{i}(2\dot{g}_{(k)j}^{i} + X^{j}\Delta_{m}[\dot{g}_{m[k]}^{i}]}, \]

therefore by similar calculation as above we can obtain:

**THEOREM 3.2.** If the tensor \( R_{jkh}^{i} \) is R-recurrent and \( H_{jkh}^{i} \) is H-recurrent, then the necessary and sufficient condition for \( M_{jkh}^{i} \) to be R-recurrent is given by:

\[ (\Delta_{m}H_{jkh}^{i})(\partial_{j}X^{m} - \Delta_{j}G^{m}) + H_{km}^{i}\dot{g}_{mj} - H_{kh}^{i}\dot{g}_{mj} - H_{nh}^{i}\dot{g}_{kj} \\
- H_{nh}^{i}\dot{g}_{mj} - H_{jkm}^{i}\dot{g}_{hl} - H_{jkm}^{i}G_{ml}^{i} + H_{nh}^{i}G_{jl}^{i} \\
+ H_{jkm}^{i}G_{jl}^{i} + H_{jkm}^{i}G_{ml}^{i} = 0. \]
Since we know that [5]:

\[ (3.7) \quad \hat{R}^{i}_{jkh} = R^{i}_{jkh} + Y_{i} \Delta^{i} R^{i}_{jkh} + L^{i}_{jkh}, \]

where

\[ (3.8) \quad L^{i}_{jkh} = 2 \left[ \partial_{\lambda} \left( \Delta^{i} \Delta^{i}_{jkh} \right) + \Delta^{i}_{jkh} \left( \partial_{\lambda} Y_{i} \right) \right. \]

\[ \left. - \left( \Delta^{i} R^{i}_{jkh} \right) \left( \partial_{\lambda} Y_{i} \right) - Y_{i} \left( \Delta^{i} R^{i}_{jkh} \right) \right] + \nabla_{\lambda} \left( \partial_{\lambda} X^{m} \right) - \nabla_{\lambda} \left( \Delta^{i} \nabla_{\lambda} \left( \partial_{\lambda} Y_{i} \right) \right). \]

therefore differentiating (3.7) covariantly with respect to \( x^{r} \) we obtain

\[ (3.9) \quad \hat{R}^{i}_{jkh, r} = R^{i}_{jkh, r} + Y_{i, r} \Delta^{i} R^{i}_{jkh} + Y_{i} \left( \Delta^{i} R^{i}_{jkh, r} \right) + L^{i}_{jkh, r,}, \]

which by virtue of equations (2.1a) and (2.1b) leads to;

**Theorem 3.3.** If \( R^{i}_{jkh} \) and \( \hat{R}^{i}_{jkh} \) are \( R \)-recurrent, then the necessary and sufficient condition for \( L^{i}_{jkh} \) to be \( R \)-recurrent is given by

\[ Y_{i, r} \Delta^{i} R^{i}_{jkh} + Y_{i} \left( \Delta^{i} R^{i}_{jkh} \right) \Delta^{i} \left( \partial_{r} X^{m} \right) \]

\[ - \Delta^{i} \left( \Delta^{i} \Delta^{i}_{jkh} \right) \nabla_{\lambda} Y_{i} - \Delta^{i} \left( \Delta^{i} \Delta^{i}_{jkh} \right) \nabla_{\lambda} Y_{i} - \Delta^{i} \left( \Delta^{i} \Delta^{i}_{jkh} \right) \nabla_{\lambda} Y_{i} \]

\[ + R^{i}_{jkh} \Delta^{i} \Delta^{i}_{jkh} + R^{i}_{jkh} \Delta^{i} \Delta^{i}_{jkh} + \nabla_{\lambda} \left( \Delta^{i} R^{i}_{jkh} \right) = 0. \]

Further from equation (3.9) one can easily establish;

**Theorem 3.4.** The necessary and sufficient condition for both \( R^{i}_{jkh} \) and \( \hat{R}^{i}_{jkh} \) to be \( R \)-recurrent is given by

\[ Y_{i, r} \Delta^{i} R^{i}_{jkh} + Y_{i} \left( \Delta^{i} R^{i}_{jkh} \right) \Delta^{i} \left( \partial_{r} X^{m} \right) \]

\[ - \Delta^{i} \left( \Delta^{i} R^{i}_{jkh} \right) \nabla_{\lambda} Y_{i} - \Delta^{i} \left( \Delta^{i} R^{i}_{jkh} \right) \nabla_{\lambda} Y_{i} - \Delta^{i} \left( \Delta^{i} \Delta^{i}_{jkh} \right) \nabla_{\lambda} Y_{i} \]

\[ + R^{i}_{jkh} \Delta^{i} \Delta^{i}_{jkh} + R^{i}_{jkh} \Delta^{i} \Delta^{i}_{jkh} + \nabla_{\lambda} \left( \Delta^{i} R^{i}_{jkh} \right) \]

\[ + L^{i}_{jkh, r} - \lambda_{r} L^{i}_{jkh} = 0. \]
DEFINITION 3.1. If in a Finsler space $F_n$, non-linear connection coefficient $\Gamma_{ik}^j$ is independent of $X^i$, then it will be called a *generalised affinely connected space* and will be abbreviated as GAC-space.

From the above definition we can observe that $F_n$ is a GAC-space if it satisfies

\begin{equation}
\Delta \Gamma_{ik}^j = 0,
\end{equation}

which by virtue of (1.5) implies that

\begin{equation}
\Delta G_{ik} = 0.
\end{equation}

Hence;

**THEOREM 3.5.** Every GAC-space $F_n$ is affinely connected but the converse is not true.

Further from equation (1.5) for a GAC-space $F_n$ one can easily obtain

\begin{equation}
\Gamma_{ik}^j + \Gamma_{jk}^i = 2G_{ij}^k,
\end{equation}

which together with (1.6) and (3.11) leads to

\begin{equation}
\Gamma_{ik}^j + \Gamma_{jk}^i = 2\Gamma_{ik}^j.
\end{equation}

where $\Gamma_{ik}^j$ is Cartan's coefficient of connection [7].

In case of a GAC-space $F_n$, one can easily establish

\begin{equation}
g_{tr,ij} = 0,
\end{equation}

which together with theorem (2.3) implies;

**THEOREM 3.6.** For a GAC-space $F_n$, if $R_{jh}^i (R_{jhh})$ is $R$-recurrent then $R_{jh}^i (R_{jhh})$ is also $R$-recurrent and conversely.

**REMARKS.** i) A theorem similar to above follows from theorem (2.4) also.

ii) In case $\Gamma_{ik}^j$ is symmetric and the space $F_n$ is affinely connected we can observe that $g_{tr,ij} = 0$. Thus theorem (3.6) can also be obtained alternatively.

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