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GENERIC SUBMANIFOLDS WITH SEMIDEFINITE SECOND FUNDAMENTAL FORM OF A COMPLEX PROJECTIVE SPACE

By Kentaro Yano and Masahiro Kon

0. Introduction

Let CP^{n+p} denote the complex projective space of real dimension $n+p$ (complex dimension $(n+p)/2$) with constant holomorphic sectional curvature 4. We denote by J the almost complex structure tensor field of CP^{n+p}. Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^{n+p}.

We denote by g the Riemannian metric tensor field induced on M from that of CP^{n+p}.

When the transform of the normal space $T_x(M)^{\perp}$ at x of M by J is always tangent to M, that is, $JT_x(M)^{\perp}\subset T_x(M)$ for any $x, T_x(M)$ being the taugent space at x of M, the submanifold M is said to be *generic* in CP^{n+p}. If M is a real hypersurface of CP^{n+1}, then M is obviously a generic submanifold.

In [1], Okumura proved the following theorems:

THEOREM **A**. *Let M be a compact orientable real hypersurface of CP^{n+1} with constant mean curvature such that the second fundamental form A is semidefinite. If* $\mathrm{Tr}A^2\leq n-1$, *then* $\mathrm{Tr}A^2=n-1$ *and* $M=M^C_{p,o}$, $p=(n-1)/2$.

THEOREM **B**. *Let M be a compact orientable real hypersurface of CP^{n+1} with constant mean curvature such that the second fundamental form A is semidefinite. If* $(\mathrm{Tr}A)^2\leq(n-1)^2$, *then* $M=M^C_{p,o}$, $p=(n-1)/2$.

The purpose of the present paper is to prove generalizations of theorems A and B for generic submanifolds of CP^{n+p} with flat normal connection.

1. Preliminaries

Let M be an n-dimensional generic submanifold of CP^{n+p}. For any vector field X tangent to M, we put $JX=PX+FX$, where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$. The operator of covariant differentiation with respect to the Levi-Civita connection in CP^{n+p} (resp. M) will be denoted by $\overline{\nabla}$ (resp. ∇). The Weingarten formula is given by $\overline{\nabla}_X V = -A_V X + D_X V$ for any vector field X tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A is called the *second fundamental form* of M. For any normal vector V, A_V is a symmetric linear transformation on $T_x(M)$. Let $\{v_a\}$ be an orthonormal frame for $T_x(M)^{\perp}$. Then the *mean curvature vector* μ of M is defined to be $\mu=\sum_a \mathrm{Tr}A_a v_a$, where $A_a=A_{v_a}$. If $D\mu=0$, then μ is said to be *parallel*. For any vector X tangent to M and any vector V normal to M, if $g(A_V X, X)\leq 0$ or $g(A_V X, X)\geq 0$, then the second fundamental form A of M is said to be *semidefinite*. We now define the curvature tensor $R^{\perp}$ of the normal bundle of M by $R^{\perp}(X,Y)=[D_X, D_Y]-D_{[X,Y]}$. If $R^{\perp}$ vanishes identically, then the normal connection of M is said to be *flat*. If the normal connection of M is flat, we can choose an orthonormal frame $\{v_a\}$ of the normal bundle such that $Dv_a=0$ for all a. Now we have

LEMMA 1. ([2]). *Let M be an n-dimensional generic submanifold of CP^{n+p} with flat normal connection. Then*

$$\sum_a \mathrm{div}\,(\nabla_{Jv_a} Jv_a)=(n-1)p-\sum_a \mathrm{Tr}A_a^2+\sum_{a,b}\mathrm{Tr}A_a g(A_a Jv_b, Jv_b)+\frac{1}{2}\sum_a |[P, A_a]|^2$$

where $[P,A_a]=PA_a-A_aP$ *and* $|T|$ *denotes the length of the tensor* T.

LEMMA 2. ([2]). *Let M be an n-dimensional generic submanifold of CP^{n+p} with flat normal connection. If the mean curvature vector of M is parallel, then*

$$g(\nabla^2 A, A)=(n-3)\sum_a \mathrm{Tr}A_a^2-\sum_a(\mathrm{Tr}A_a)^2+3\sum_a |[P, A_a]|^2+2p(p-1)$$
$$+\sum_{a,b}[3g(A_aJv_b, Jv_b)\mathrm{Tr}A_a-(\mathrm{Tr}A_aA_b)^2+(\mathrm{Tr}A_a)(\mathrm{Tr}A_b^2A_a)].$$

Model space: Let S^{n+2} be sphere with radius 1. In S^{n+2} we have the family of generalized Clifford surfaces $M_{p,q}=S^p(r)\times S^q(r)$, $r_1^2+r_2^2=1$, $p+q=n+1$. By choosing the spheres in such a way that they lie in complex subspaces we get fibrations $S^1\longrightarrow M_{2p+1,2q+1}\stackrel{\pi}{\longrightarrow} M^C_{p,q}$ compatible with Hopf fibration, where $p+q=(n-1)/2$. Thus we see that $M^C_{p,q}=\pi(S^{2p+1}(r_1)\times S^{2q+1}(r_2))$, $r_1^2+r_2^2=1$. In the special case $q=0$, $M^C_{p,0}$ is called the *geodesic hypersphere*.

2. Theorems

First of all, we prove

THEOREM 1. *Let M be a compact orientable n-dimensional generic submanifold of* CP^{n+p} *with flat normal connection such that the second fundamental form is semidefinite. If* $\sum_a \mathrm{Tr}A_a^2 \leq (n-1)p$, *then* $p=1$ *and M is the geodesic hypersphere* $\pi(S^n(r)\times S^1(r)), r=(1/2)^{1/2}$, *of* CP^{n+1}.

PROOF. Since M is compact orientable, lemma 1 implies that

$$\int_M [(n-1)p-\sum_a \mathrm{Tr}A_a^2+\sum_{a,b} \mathrm{Tr}A_a g(A_aJv_b, Jv_b)]*1 = -\frac{1}{2}\int_M \sum_a |[P, A_a]|^2*1.$$

From the assumptions we see that the left hand side of this equation is nonnegative. Thus we obtain $\sum_a \mathrm{Tr}A_a^2=(n-1)p$, $PA_a=A_aP$ and $\mathrm{Tr}A_a g(A_aJv_b, Jv_b)=0$ for any a and b. Suppose that $\mathrm{Tr}A_a=0$ for some a. Since the second fundamental form is semidefinite, we see that $A_a=0$. On the other hand, the equation of Codazzi is given by

$$(\nabla_X A)_V Y-(\nabla_Y A)_V X=-g(X, JV)PY+g(JV, Y)PX-2g(X, PY)JV.$$

Putting $V=v_a$ and $X=JV$ in this equation, we obtain $g(JV, JV)PY=0$. Thus we have $P=0$. Then M is anti-invariant and $\sum_b g(A_aJv_b, Jv_b)=\mathrm{Tr}A_a$ for any a. Therefore $\mathrm{Tr}A_a g(A_cJv_b, Jv_b)=0$ implies that $\mathrm{Tr}A_a=0$ for all a and hence M is totally geodesic. This contradicts to the fact that $\sum_a \mathrm{Tr}A_a^2=(n-1)p$. Consequently, we must have $\mathrm{Tr}A_a\neq 0$ for all a and then $g(A_cJv_b, Jv_b)=0$ for any a and b. Let V be a unit vector normal to M. We take an orthonormal frame $\{V_a\}$ of $T_x(M)^\perp$ such that $V=V_1$. Then we obtain

$$\begin{aligned}\sum_d g(A_aJV_d, JV_d)&=\sum_{b,c,d} g(V_d, v_b)g(V_d, v_c)g(A_aJv_b, Jv_c)\\&=\sum_b g(A_aJv_b, Jv_b)=0.\end{aligned}$$

From this we see that $g(A_aJV, JV)=0$ for any unit vector V normal to M. Since A_a is symmetic, we obtain $g(A_aJv_b, Jv_c)=0$ for any a, b and c by putting $V=(v_a+v_c)/\sqrt{2}$. We now use the following equation of Ricci

$$g(R^\perp(X, Y)U, V)+g([A_V, A_U]X, Y)=g(FY, U)g(FX, V)-g(FX, U)g(FY, V).$$

Putting $V=v_a, U=v_b, X=Jv_a$ *and* $Y=Jv_b$ in this equation, and using $R^\perp=0$, we have $g(v_b, v_b)g(v_a, v_a)-g(v_a, v_b)^2=0$. Therefore we obtain $p=1$. On the other hand, $PA_a=A_aP$ implies that the mean curvature of M is constant because of $p=1$ (see [1]). Thus our theorem follows from theorem A.

From theorem 1 we have

THEOREM 2. *Let M be a compact orientable real hypersurface of CP^{n+1} such that the second fundamental form is semidefinite. If* $\mathrm{Tr}A^2 \leq n-1$, *then M is the geodesic hypersphere* $\pi(S^n(r)\times S^1(r)), r=(1/2)^{1/2}$.

THEOREM 3. *Let M be a compact orientable n-dimensional generic submanifold of CP^{n+p} with flat normal connection such that the second fundamental form is semidefinite. If the mean curvature vector of M is parallel and* $\sum_a(\mathrm{Tr}A_a)^2 \leq (n-1)^2 p$, *then $p=1$ and M is the geodesic hypersphere* $\pi(S^n(r)\times S^1(r))$, $r=(1/2)^{1/2}$, *of* CP^{n+1}.

PROOF. Since M is compact orientable, lemma 2 implies that

$$\begin{aligned}\int_M [|\nabla A|^2 &-2(n-p)p+3\sum_a |[P, A_a]|^2]*1\\ =\int_M [&\sum_{a,b}(\mathrm{Tr}A_aA_b)^2-\sum_{a,b}(\mathrm{Tr}A_a)\ (\mathrm{Tr}A_b^2A_a)-(n-3)\sum_a \mathrm{Tr}A_a^2\\ &+\sum_a(\mathrm{Tr}A_a)^2-3\sum_{a,b}\mathrm{Tr}A_a g(A_aJv_b, Jv_b)-2(n-1)p]*1.\end{aligned}$$

For any a, we put $K_aX=A_aX+(\mathrm{Tr}A_a g(Jv_b, X)Jv_b)/(n-1)$ for some b. Since K_a is symmetric, we see that $n\mathrm{Tr}K_a^2 \geq (\mathrm{Tr}k_a)^2$. From this we have

$$(\mathrm{Tr}A_a)^2 \leq (n-1)\mathrm{Tr}A_a^2+2\mathrm{Tr}A_a g(A_aJv_b, Jv_b)$$

for any a and b. Thus we obtain

$$\sum_a(\mathrm{Tr}A_a)^2 \leq (n-1)\sum_a \mathrm{Tr}A_a^2+2\sum_a \mathrm{Tr}A_a g(A_a Jv_b, Jv_b).$$

Therefore we see that

$$\begin{aligned}\int_M [|\nabla A|^2-2(n-p)p+3\sum_a |[P, A_a]|^2]*1\\ \leq\int_M [\sum_{a,b}(\mathrm{Tr}A_aA_b)^2-\sum_{a,b}(\mathrm{Tr}A_a)(\mathrm{Tr}A_b^2A_a)+\frac{2}{n-1}\{\sum_a(\mathrm{Tr}A_a)^2\\ -(n-1)^2p\}-\frac{n+3}{n-1}\sum_{a,b}\mathrm{Tr}A_a g(A_a Jv_b, Jv_b)]*1.\end{aligned}$$

On the other hand, we have $|\nabla A|^2 \geq 2(n-p)p$ (see [2]) and hence the left hand side of this inequality is nonnegative. Moreover, from the assumption we see that the right hand side of this inequality is nonpositive. Consequently, we have $\sum_a(\mathrm{Tr}A_a)^2=(n-1)^2p$, $(\mathrm{Tr}A_aA_b)^2 = (\mathrm{Tr}A_a)(\mathrm{Tr}A_b^2A_a)$ and $\mathrm{Tr}A_a g(A_aJv_b, Jv_b)=0$ for any a and b. By a similar method as that used in the proof of theorem 1 we see that $p=1$. Therefore our theorem follows from theorem B.

Tokyo Institute of Technology
and
Hirosaki University

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